London School of Geometry and Number Theory

Palindromic Bruhat Ideals and Hyperplane Arrangements

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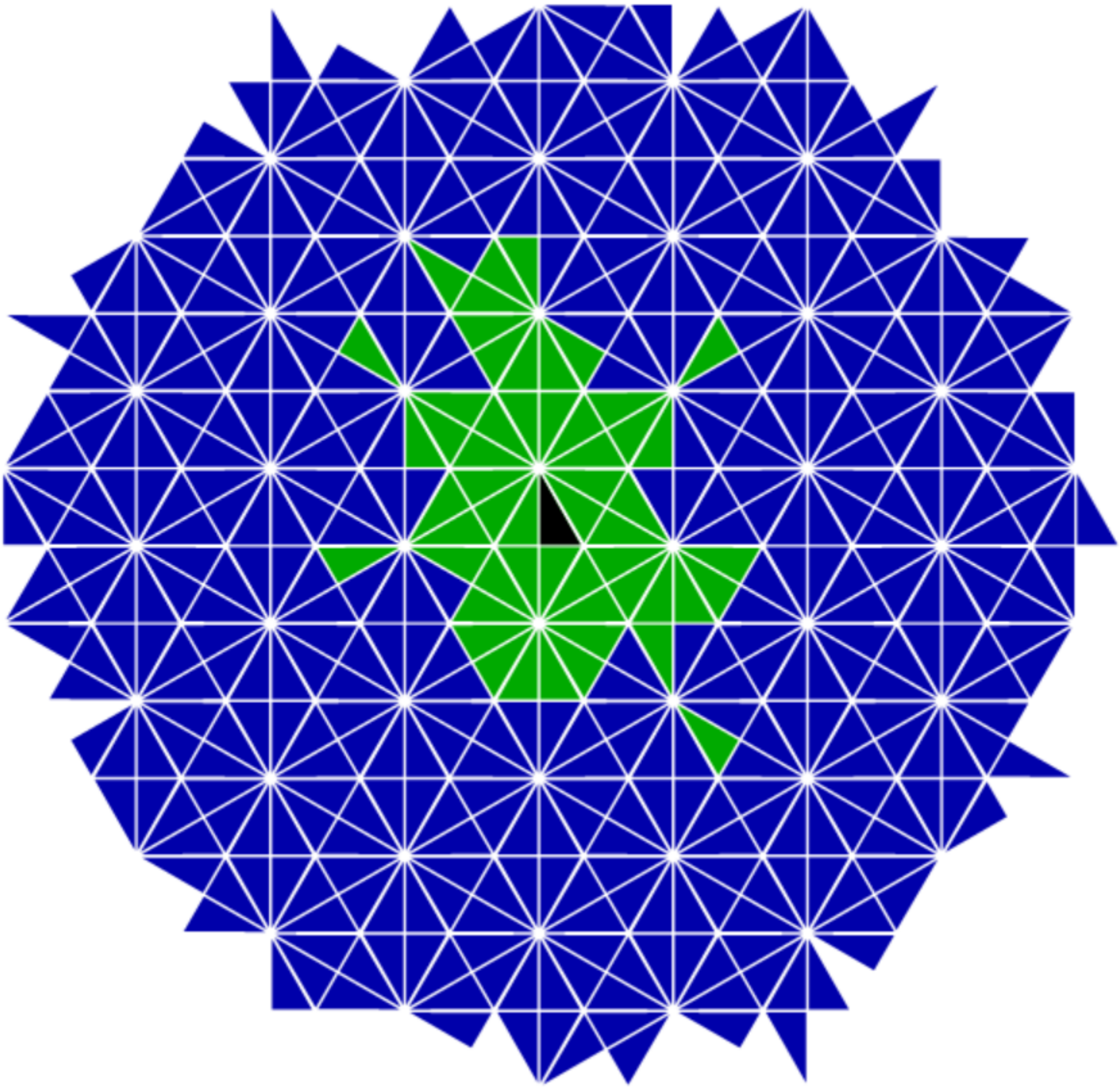

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Abstract

It is well known that the Schubert varieties in a flag manifold (important examples of algebraic varieties), are indexed by elements of a Weyl group; and moreover the containment relations between these varieties is given by the Bruhat order on the Weyl Group. In [17] the authors characterise the smoothness of Schubert varieties in Grassmannians by looking at certain hyperplane arrangements associated to the Weyl group elements, and in [18] they extend this to generalised flag manifolds. In this report we sketch the link between Schubert varieties, Weyl groups, and Bruhat order; explore the characterisations mentioned above; and discuss a conjecture of S. Oh and H. Yoo extending their work to arbitrary Coxeter groups. Our thanks go to Professor Konstanze Rietsch for her help supervising this project.

Contents

I Introduction

This introductory chapter covers the basics of Coxeter groups and Schubert varieties, and how they are related via Bruhat ordering. The rest of the report is concerned with characterising smoothness of Schubert varieties by looking at the geometry of the associated Weyl group. For this reason the geometry of Coxeter groups is emphasised, and the details of the algebraic geometry of Schubert varieties is only sketched. In particular we will not actually be using the formal definition of (rational) smoothness, so we opt instead to define this by an equivalent combinatorial property which is much easier to work with for our purposes. The last section of this chapter surveys some of the previously known results on smoothness of Schubert varieties, in particular results about pattern avoidance.

I.1 Coxeter Groups and Weyl Groups

Coxeter groups are groups generated by reflections which act discretely. They arise naturally as the symmetry groups of regular polyhedra, for example the symmetry group of the regular tetrahedron is S_4, generated by the transpositions $(i, i+1)$ which act as reflections. Another common source of examples is periodic tilings of Euclidean space, for example the tiling of the plane by squares, equilateral triangles, or hexagons. Initially we shall introduce Coxeter groups in terms of generators and relations, however it can be shown using some of the tools mentioned in Section 1C that all discrete reflection groups admit a presentation of the stated form, and vice versa.

1A Basic Definitions

Definition I.1. A group W is a **Coxeter group** if it admits a presentation of the form

$$\langle s_1, \ldots, s_k \mid (s_i s_j)^{m_{ij}} = e \rangle$$

where $S = \{s_1, \ldots, s_k\}$ is a (finite) set of generators for W, and $m_{ij} \in \{1, 2, \ldots, \infty\}$ is symmetric in its indices and satisfies $m_{ij} = 1$ if and only if $i = j$. We call the pair (W, S) a **Coxeter system** for W, and the generators S are called the **simple reflections** of (W, S); k is the **rank** of W. The set of **reflections** of the Coxeter system (W, S) is the set $R(W, S)$ of all conjugates of the simple reflections.

Every element in the set $R(W, S)$ (which contains S) has order 2 in W, as one would expect from a geometric reflection. In general the Coxeter presentation of a Coxeter group W is not unique, and indeed W may have different Coxeter presentations which may not correspond under automorphisms of the group, or even have the same number of generators, and this is why we talk about Coxeter *systems* instead.

All of the information needed to reconstruct W is given by the numbers m_{ij} for $i \neq j$. It is common to summarise this data in a graph called the **Coxeter diagram** (which is closely related to, and sometimes called, the Dynkin diagram). This graph has vertex set S, and an edge between s_i and s_j if $m_{ij} > 2$. These edges are labelled with the corresponding value m_{ij}, although edge labels 3 are typically omitted. Note that generators s_i and s_j commute if and only if $m_{ij} = 2$, or equivalently if they are not joined by an edge in the Coxeter diagram of (W, S).

Definition I.2. Let (W, S) be a Coxeter system, and suppose $S = T \cup T'$ for non-empty disjoint sets T and T', such that if $s_i \in T$ and $s_j \in T'$ then $m_{ij} = 2$ (i.e. the Coxeter diagram of (W, S) is not connected, and T and T' correspond to components of the graph), then $W = \langle T \rangle \times \langle T' \rangle$ and we say W is **reducible**. If no such decomposition is possible, then W is **irreducible**.

For any subset $T \subset S$, $(\langle T \rangle, T)$ is itself a Coxeter system, we shall write $W_T = \langle T \rangle$.

Definition I.3. Let (W, S) be a Coxeter system, and let $w \in W$. We say a finite sequence of letters $w = t_1 \cdots t_d$ is a **word** representing w, for $t_i \in S$. The **length function** for (W, S) is $l_S : W \mapsto \mathbb{N}$ given by $l_S(w) := \min\{d \mid w = t_1 \cdots t_d\}$. If S is clear from context we shall just write $l(w)$. If $t_1 \cdots t_d$ is a minimal length word representing w, then we call it **reduced**.

For the following chapters there is an important collection of reflections associated to each element of a Coxeter group.

Definition I.4. Let (W, S) be a Coxeter system, and let $t_1 \cdots t_d$ be a fixed reduced word for $w \in W$. For each $1 \leq i \leq d$, let $p_i = t_1 \cdots t_{i-1} t_i t_{i-1} \cdots t_1$. We call p_i an **inversion** of w, and write $\mathrm{inv}(w) = \{p_i\}_{i=1}^d$ for the set of inversions of w. We shall see later that $\mathrm{inv}(w)$ does not depend on the choice of reduced word for w (Proposition I.2).

The following is a very important property (in fact characterisation) of Coxeter systems.

Theorem I.1 (Strong Exchange Condition)**.** *[5, Theorem 1.4.3] Let (W, S) be a Coxeter system with $w \in W$. Let $t_1 \cdots t_d$ be an expression for w, and let $r \in R(W, S)$. If $l(rw) < l(w)$, then there is some index i such that $rw = t_1 \cdots \hat{t}_i \cdots t_d$ (where a hat denotes that that letter has been deleted).*

1B Bruhat Order

Central to our discussion will be a partial ordering of the elements of a Coxeter system related to the length function and to the reflections of a Coxeter system. Our main source for this section is [5].

Definition I.5. Let (W, S) be a Coxeter system, the **Bruhat order** on W is defined as follows: let $u, v \in W$ and $r \in R(W, S)$

1. We write $u \xrightarrow{r} v$ if $u = rv$ and $l(u) < l(v)$.
2. We write $u \to v$ if $u \xrightarrow{r} v$ for some $r \in R(W, S)$.
3. We define $u \leq v$ if there is a sequence of elements $\{u_1, \ldots, u_k\}$ such that

$$u = u_1 \to u_2 \to \cdots \to u_{k-1} \to u_k = v.$$

Given two group elements $u, v \in W$, the **Bruhat interval** $[u, v]$ is the set of elements $x \in W$ such that $u \leq x \leq v$. Here we shall be interested in the **Bruhat ideal** $[e, w]$ for some fixed $w \in W$. The **Bruhat graph** of $w \in W$ is the directed graph whose vertex set is $[e, w]$, and whose edges are given by the relations $u \to v$ defined above. The **Bruhat covering graph** is the subgraph containing only those edges $u \to v$ for which $l(v) - l(u) = 1$.

It is clear that if $u \leq v$ then $l(u) \leq l(v)$, and in fact Bruhat intervals are posets graded by the length function.

Proposition I.1. *Let (W, S) be a Coxeter system, $w \in W$, and $r \in R(W, S)$, then $l(rw) < l(w)$ if and only if $r \in \mathrm{inv}(w)$.*

Proof. First let $r \in \mathrm{inv}(w)$ and let $w = t_1 \cdots t_d$ be a reduced word, then by definition there is $1 \leq i \leq d$ such that $r = p_i$. Hence

$$rw = t_1 \cdots t_{i-1} t_i t_{i-1} \cdots t_1 t_1 \cdots t_d = t_1 \cdots \hat{t}_i \cdots t_d$$

in particular $l(rw) \leq d = l(w)$.

Conversely suppose r shortens w, so that we can apply the strong exchange condition, which says that there is and index i such that $rw = t_1 \cdots \hat{t}_i \cdots t_d$. Multiplying on the right by $w^{-1} = t_d \cdots t_1$ gives

$$r = rww^{-1} = t_1 \cdots t_{i-1} t_i t_{i-1} \cdots t_1 = p_i$$

so $r \in \mathrm{inv}(w)$. ■

This says that if $u \to v$, then there are words representing u and v such that u is obtained by deleting a letter in the word for v. The following property of Bruhat order now follows straightforwardly from the definition.

Theorem I.2 (Subword Property)**.** *[5, Corollary 2.2.3] Let (W,S) be a Coxeter system, and $u,v \in W$, then then following are equivalent*

1. *$u \leq v$,*
2. *Every reduced word for v has a subword which is a reduced word for u, and*
3. *Some reduced word for v has a subword which is a reduced word for u.*

Remark I.1. The definition of Bruhat order appears to be asymmetric in that we insist on multiplying on the *left* by the reflection r, however an almost identical proof shows that the subword property also holds for the "right-handed" version of Bruhat order, and so these two definitions coincide. It is also clear from this characterisation that $[e,w]$ is always a finite set.

1C Geometry

Jacques Tits, who developed much of the classical theory of Coxeter groups in the 1960s, constructed the so-called reflection representation for an arbitrary Coxeter system. While the definition is very straightforward (and can be found in most texts on Coxeter groups) we shall omit the details of its construction here, and focus on its properties. Much of the material in this section can be found in the first few chapters of [8]. Given any Coxeter system, one can define a symmetric bilinear form B on a real vector space V whose dimension is the rank of (W,S). The representation makes W act on V such that the elements of $R(W,S)$ act as orthogonal (with respect to B) reflections with respect to hyperplanes in V (i.e. codimension 1 linear *subspaces*). Throughout this report we shall denote this hyperplane arrangement canonically associated to W by $\mathcal{H}$. Furthermore this representation is faithful, and the action is discrete in the interior of a cone in V called the Tits cone.

The set of hyperplanes in V associated to W divide up the interior of the Tits cone into regions called the **chambers** of (W,S). The hyperplanes corresponding to the generators S define a chamber which we shall call the **fundamental chamber**. W acts simply transitively on the chambers, so given any chamber, we can label it by the unique element of W which maps the fundamental chamber to the given chamber. This encodes not only the group elements in the geometry, but also the combinatorial structure, as we shall see below.

Let us suppose conversely that we have a group W acting on, for example, a sphere, Euclidean space, or hyperbolic space; such that the action is discrete, and is generated by reflections in geodesic hyperplanes. Take one of the connected components of the space once all these hyperplanes are removed, and call it the the fundamental chamber. Let $S = \{s_1, \dots, s_k\}$ be the reflections in the hyperplanes which define the fundamental chamber. Then (W,S) is a Coxeter system. Moreover if we construct the reflection representation of (W,S), although we will not get exactly back to the picture we started with, the essential geometric features will be unchanged. In particular the "shape" of the chambers up to taking cross products with copies of $\mathbb{R}^n$, their adjacency relations, number of walls, the angles at which hyperplanes intersect, and so on. In the particular cases of the examples mentioned at the start of the paragraph, almost exactly the original picture can be recovered by intersecting the Tits cone with respectively a sphere, an affine hyperplane of appropriate codimension, or a hyperboloid model of hyperbolic space.

The main part of this report is engaged in trying to characterise combinatorial properties of (W,S), and in particular of the Bruhat order, by geometric properties of sub-hyperplane arrangements in the reflection representation of (W,S), so in the remainder of this section we shall reinterpret the algebraic and combinatorial definitions given thus far in terms of the geometry of the reflection representation. As mentioned above the chambers correspond to the group elements, and the set of hyperplanes $\mathcal{H}$ corresponds to the set of reflections $R(W,S)$.

Definition I.6. Let C be a chamber, a **wall** of C is a hyperplane $H \in \mathcal{H}$ such that $\dim(\overline{C} \cap H) = \dim(H)$, and in this case $\overline{C} \cap H$ is called a **face** of C. Two chambers C and D are **adjacent** if they share a common wall (note that we do not exclude the possibility that $C = D$). A **gallery** Γ is a sequence of chambers $(C_1, \dots, C_d)$ such that consecutive chambers are adjacent. Γ is said to **stutter** if $C_i = C_{i+1}$ for some i. We can define a **distance function** on the set of chambers by letting $d(C,D)$ be the minimum over the lengths of all galleries joining C and D.

It is clear that any gallery joining chambers C and D must cross every hyperplane which separates them, and hence $d(C,D)$ is equal to the number of hyperplanes which separate them. What is remarkable is the connection between (reduced) words representing w in W, and (minimal) galleries in the reflection representation.

Lemma I.1. *[8, Theorem I.5A] Let C be the fundamental chamber (i.e. the chamber labelled with the identity), and wC the chamber labelled by w. Let $t_1, \ldots, t_d \in S$, then $t_1 \cdots t_d$ is a (reduced) word representing w if and only if*

$$\Gamma = (C,\ t_1C,\ t_1t_2C, \ldots,\ t_1 \cdots t_{d-1}C,\ t_1 \cdots t_{d-1}t_dC)$$

is a (minimal) gallery joining $C = eC$ and $t_1 \cdots t_dC = wC$.

This lemma means that $d(C, wC) = l(w)$. Moreover, we can label the faces of C by the elements of S, by saying $\overline{C} \cap H$ has label $s \in S$ if C and sC are adjacent across H. By simple transitivity, this labelling can be extended in a well-defined way to the faces of all the chambers. Then if D is a chamber, D and sD are adjacent, and share a face labelled by s. The inversions of an element w correspond to a special subset of the hyperplanes $\mathcal{H}$.

Proposition I.2. *Let (W, S) be a Coxeter system, and $w \in W$. Let $r \in R(W, S)$ and $H \in \mathcal{H}$ be the corresponding hyperplane. Let C be the fundamental chamber. Then $l(rw) < l(w)$ if and only if H separates the chambers C and wC.*

Proof. Assume $l(rw) < l(w)$, H necessarily separates wC and rwC, then there are two possibilities:

$$C, wC \underset{H}{|} rwC \qquad\qquad C, rwC \underset{H}{|} wC$$

where this notation means that, for example in the first case, H separates C and wC from rwC. In the second case we are done, so assume we are in the first situation for a contradiction. Let Γ be a minimal gallery from C to rwC which necessarily crosses H, then reflect the part of the gallery in the half-space with respect to H which contains rwC to give a new gallery Γ' of the same length from C to wC. In Γ there were exactly two chambers which had H as a wall, call these D and D', so we had

$$\Gamma = (C, \ldots, D \underset{H}{|} D', \ldots, rwC)$$

In Γ', D is left fixed, but D' is reflected in H onto D, so Γ' stutters at D. One of these D's can be deleted to give a new shorter gallery Γ'' from C to wC. Since we assumed Γ was minimal, this means $l(w) < l(rw)$, which contradicts our earlier assumption.

Conversely, assume H separates C and wC, and let Γ be a minimal gallery from C to wC which necessarily crosses H. As above we can construct a gallery Γ' from C to rwC by reflecting the chambers beyond H, in H. This gallery stutters, and so can be shortened in the same way to give a shorter gallery Γ'' from C to rwC. This means that $l(rw) < l(w)$ as required. ∎

Together with Proposition I.1, we have characterised the inversions of w as reflections whose corresponding hyperplane separates C and wC in the reflection representation. This also proves the earlier claim that the set $\mathrm{inv}(w)$ does not depend on the choice or reduced word representing w.

Remark I.2. Taking the previous two results together we see that $w \in W$ has a reduced word starting with $s \in S$ if and only if $s \in \mathrm{inv}(w)$. This motivates us to define $D(w) = S \cap \mathrm{inv}(w)$, which (following [5]) we shall call the **descent set** of w.

We can also interpret Bruhat ideal geometrically. Let D be a chamber, and say we **invert** D if we reflect it in a hyperplane which separates it from the fundamental chamber. Then the Bruhat ideal $[e, w]$ is the smallest collection of chambers which contains wC and is closed under inversion. It follows easily that closure of this collection of chambers forms a simply-connected set (consider inverting chambers in their walls which separate them from C).

1D Finite Coxeter Groups and Weyl Groups

Large classes of Coxeter groups have been classified, starting with the finite Coxeter groups by H.S.M. Coxeter in 1935. It can be shown that a Coxeter group is finite if and only if the symmetric bilinear form B mentioned above is positive definite. In this case the reflection representation acts discretely on the whole ov V, and stabilises a codimension 1 sphere centred on the origin; for this reason finite Coxeter groups are often referred to a *spherical.* There are 4 infinite families, and 4 finite families, each family is given a letter (its **type**), and then a subscript number gives the rank of the group. The classification is given in terms of Coxeter diagrams in Table I.1, which lists all finite irreducible Coxeter groups. All finite Coxeter groups decompose as a Cartesian product of copies of these groups. Similar tables can be found for Euclidean and hyperbolic Coxeter groups.

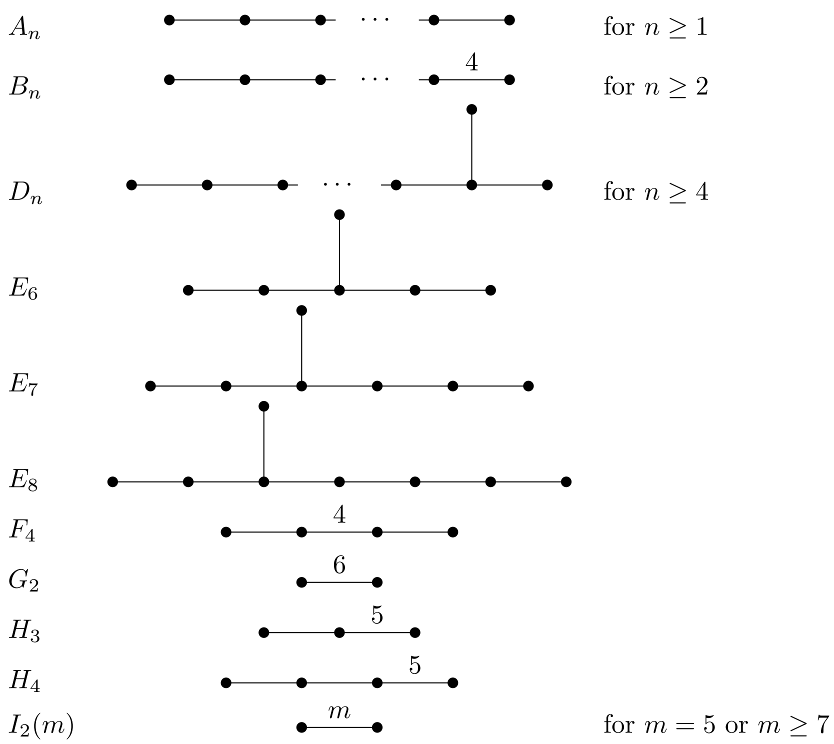


Table I.1: All Coxeter diagrams corresponding to finite irreducible Coxeter systems.

It is a well-known property of finite Coxeter groups that they have a unique longest element, usually denoted w_0. This element is important for studying the group as a whole, in particular $\mathrm{inv}(w_0) = R(W, S)$ so $\#R(W, S) = l(w_0)$. It is also the case that $[e, w_0] = W$; we shall use this in Chapter III.

One way to study Coxeter groups is by looking at their associated hyperplanes and chambers as above. A dual perspective is to consider the associated root system. This material is treated in, for example, [5].

Definition I.7. Let V be a real finite dimensional vector space. A subset Φ is a **root system** if it satisfies:

1. Φ is finite, does not contain the zero vector, and spans V,
2. If $c \in \mathbb{R}$ is such that $\alpha, c\alpha \in \Phi$, then $c = \pm 1$, and
3. For each $\alpha \in \Phi$, let r_α be the orthogonal reflection in $\alpha^\perp$, then $r\Phi = \Phi$.

A root system is **crystallographic** if for any $\alpha, \beta \in \Phi$, $r_\alpha\beta - \beta$ is an integer multiple of α.

It is clear that every finite Coxeter group W has an associated root system: take the closure under the action of W of a set of unit normals of the hyperplanes corresponding to the simple reflections. Conversely any root system gives rise to a finite Coxeter group denoted $W(\Phi)$. If Φ is a crystallographic root system, then the corresponding Coxeter group is called a **Weyl group**. Not all finite Coxeter groups arise as Weyl groups; the types which do are A, B, D, E, F, G.

I.2 Flag Manifolds and Schubert Varieties

This section introduces a completely different theme, that of flag manifolds, which are important examples of algebraic varieties, and which have close connections with Lie groups. In Section 2B we connect with the preceding material by drawing the link between flag manifolds, Weyl groups, and Bruhat order. We follow [2, Chapters 1 and 2] in the main for this section; see also [22], and [5, Example 1.2.11].

2A Flag Manifolds

Throughout let G be an **algebraic group**, i.e. a group which has the structure of an algebraic variety over an algebraically closed field $\mathbb{F}$, which is compatible with the group law and taking inverses. A **torus** in G is an algebraic subgroup which is isomorphic to a product of copies of $\mathbb{F}^*$. All maximal tori in G are conjugate, and so have the same dimension l which we call the **rank** of G. A **Borel subgroup** of G is a maximal connected solvable subgroup B; all Borel subgroups are conjugate, so these too all have the same dimension. The **radical** of G, written $R(G)$, is the connected component of the intersection of all the Borel subgroups of G which contains the identity. We say that G is **semi-simple** if $R(G)$ is trivial. Let us assume that G is a semi-simple, simply connected algebraic group, B a Borel subgroup, and T a maximal torus such that $T \subset B \subset G$.

Given such a set-up, we can associate a crystallographic root system to G, and hence a Weyl group. Let $V = \mathfrak{g} =: \mathrm{Lie}(G)$, on which T acts via the adjoint representation. Write $\Psi(T) = \mathrm{Hom}_{\text{alg. gp.}}(T, \mathbb{F}^*)$ for the characters of T, then we say $\chi \in \Psi(T)$ is a **weight** in V if

$$V_\chi := \{v \in V \mid tv = \chi(t)v \ \forall t \in T\}$$

is non-zero. Denote the set of weights for T in V by $\Phi(T, V)$, then this is a crystallographic root system, thought of as living in $\Psi(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $W = W(\Phi(T, V))$, this is isomorphic to $N(T)/T$ where $N(T)$ is the normaliser of T.

Example I.1. The simplest example is to take $G = SL_n(\mathbb{F})$, $B = T_n(\mathbb{F})$ the subgroup of upper triangular matrices, and T the subgroup of diagonal matrices. Then the associated Weyl group is the symmetric group on n letters, i.e. A_{n-1}.

Given G, B, T, and W as above, the associated **flag manifold** is G/B which we say has **type** the type of W (so $SL_n(\mathbb{F})/T_n(\mathbb{F})$ has type A). The set of T-fixed points in G/B is $\{wB \mid w \in W\}$. The B-orbits of these points are the double cosets BwB which partition G/B by the basic properties of double cosets. The sub-varieties of G/B obtained by taking the Zariski closures $X(w) := \overline{BwB}$ are the **Schubert varieties** of G/B, and the BwB's are called the **Schubert cells** of G/B.

2B Bruhat Decomposition

As noted above

$$G/B = \bigsqcup_{w \in W} BwB$$

This is in fact a cell decomposition in the sense of a CW-complex. Moreover, and somewhat astonishingly, the face or containment relations between Schubert cells is given by the Bruhat order on W, so

$$BuB \subset \overline{BwB} \Leftrightarrow u \leq w$$

It follows from this that the Schubert varieties also decompose as

$$X(w) = \bigsqcup_{u \leq w} BuB = \bigsqcup_{u \in [e,w]} BuB$$

I.3 Previous Results on Smoothness

A point in a variety is **smooth** or **non-singular** if the tangent space of the variety at that point has the same dimension as the variety itself, otherwise it is a **singular point**. If $X(w)$ is not smooth, the singular locus is a non-empty B-stable closed sub-variety of $X(w)$. To decide if a point $x \in X(w)$ is smooth, it suffices to check the T-fixed point in the B-orbit of x. There is a weaker notion of smoothness called **rational smoothness** which is defined using the étale cohomology of the variety, we shall omit the details of the definition because for the remainder of the text we shall use the characterisation of rational smoothness due to J. B. Carrell and D. Peterson given in Theorem I.3. Smoothness implies rational smoothness, but not the other way around. The converse does hold in certain cases: type A was proved by V. V. Deodhar in [10], and types D and E by D. Peterson, published by J. B. Carrell in [9].

3A Poincaré Polynomial

If we work over $\mathbb{C}$, $X(w)$ is rationally smooth if and only if the singular cohomology ring $H^*(X(w))$ admits Poincaré duality [16], which in turn implies that the rank generating function of $H^*(X(w))$ is a palindromic polynomial called the Poincaré polynomial of $X(w)$. It follows from Bruhat decomposition that this is equal to the length generating function of the Bruhat ideal of w, $[e, w]$, thus we make the following definition.

Definition I.8. Let (W, S) be a Coxeter system with length function l. The **Poincaré polynomial** of $w \in W$ is

$$P_w(q) = \sum_{u \leq w} q^{l(u)}$$

So this coincides with the cohomological Poincaré polynomial of the Schubert variety in the case W is a Weyl group.

We can now give the theorem-definition of rational smoothness mentioned above.

Theorem I.3. *[9, Theorem E] Let G be semi-simple, and $X(w)$ a Schubert variety in G/B, then the following are equivalent:*

1. *$X(w)$ is rationally smooth,*
2. *$P_w(q)$ is palindromic, and*
3. *The Bruhat graph of w is regular of degree $l(w)$, i.e. every vertex is incident to $l(w)$ edges.*

3B Pattern Avoidance

In the main body of this report we shall focus on geometric criteria for rational smoothness, but it is worth recording some nice, explicit, combinatorial criteria based on pattern avoidance in certain cases. Those cases are the Weyl groups of type A, B, and D. These groups all have geometric realisations (for example as the symmetry groups of the n-simplex, and the n-cube in the case of type A and B respectively), but key to pattern avoidance, they can also be interpreted combinatorially. A_{n-1} is isomorphic to the permutation group on n letters, with the i^{th} generator σ_i acting as the transposition $(i, i+1)$. B_n is isomorphic to the signed permutation group on n letters (i.e. the permutation group when the n letters have two possible states, positive or negative). The first $n-1$ generators we will denote $\sigma_1^B, \ldots, \sigma_{n-1}^B$, and they act as the generators in A_{n-1}, the final generator $\sigma_\pm^B$ has the effect of changing the sign of the first letter. D_n is an index 2 subgroup of B_n, it is the group of even signed permutations. We will denote its generators by $\sigma_1^D, \ldots, \sigma_{n-1}^D$ which as above act as the generators of A_{n-1}, and the final generator $\sigma_\pm^D$ has the effect of transposing the first two letters, and changing both of their signs.

It is convenient to write elements of A_{n-1}, B_n, and D_n in **one-line notation**. Take an element σ of one of these groups, then it is a (signed) permutation of the set $\{1, 2, \ldots, n\}$. We write σ as a string of n letters, where the i^{th} letter is

$$\begin{cases} j & \text{if } \sigma(i) = +j \\ \bar{j} & \text{if } \sigma(i) = -j \end{cases}$$

Example I.2. The element $2\bar{1}3 \in B_3$ is the signed permutation σ such that $\sigma(1) = 2$, $\sigma(2) = -1$, and $\sigma(3) = 3$. The element $213 \in A_2$ is the permutation σ such that $\sigma(1) = 2$, $\sigma(2) = 1$, and $\sigma(3) = 3$.

Definition I.9. Following the notation of S. Billey, we define the **flattening function** which takes a finite sequence of distinct non-zero integers $a_1 \cdots a_k$ and outputs the unique sequence $fl(a_1 \cdots a_k) = b_1 \cdots b_k$ satisfying:

1. For each i, $-k \leq b_i \leq k$ is a non-zero integer,
2. For each i, a_i and b_i have the same sign, and
3. For all i, j, $|b_i| < |b_j|$ if and only if $|a_i| < |b_j|$.

A sequence $a_1 \cdots a_k$ such that $fl(a_1 \cdots a_k) = a_1 \cdots a_k$ is called a **pattern**. We say that a sequence $a_1 \cdots a_k$ **avoids** the pattern $b_1 \cdots b_l$ if there is no sequence of indices $1 \leq i_1 < i_2 < \cdots < i_l \leq k$ such that $fl(a_{i_1} \cdots a_{i_l}) = b_1 \cdots b_l$.

Example I.3. Consider the sequence $6\bar{2}34\bar{8}\bar{5}$, $fl(6\bar{2}34\bar{8}\bar{5}) = 5\bar{1}23\bar{6}\bar{4}$. This sequence avoids the pattern $2\bar{1}3$, but does not avoid the pattern $21\bar{3}$ since $fl(63\bar{8}) = 21\bar{3}$.

For type A, the following theorem was proved by V. Lakshmibai and B. Sandhya.

Theorem I.4. *[15, Theorem 2.2] For an element $w \in A_n$, the Schubert variety $X(w)$ is rationally smooth if and only if w avoids both* 3412 *and* 4231*.*

This type of result was generalised to types B and D by S. Billey.

Theorem I.5. *[1, Theorems 4.2 and 6.2] For an element $w \in B_n$ or D_n, the Schubert variety $X(w)$ is rationally smooth if and only if w avoids the patterns in Table I.2.*

Bad patterns	**Type**
3412 4231	A, B, and D
$1\bar{2}\bar{3}$ $\bar{2}1\bar{3}$ $2\bar{1}\bar{3}$ $2\bar{3}\bar{1}$ $\bar{3}1\bar{2}$ $\bar{3}\bar{2}1$ $\bar{3}2\bar{1}$ $3\bar{2}\bar{1}$ $3\bar{2}1$	B
$\bar{1}2\bar{3}$ $12\bar{3}$ $1\bar{3}\bar{2}$ $\bar{2}\bar{1}\bar{3}$ $\bar{3}\bar{2}\bar{1}$ $\bar{2}\bar{4}31$ $2\bar{4}31$ $\bar{3}\bar{4}\bar{1}\bar{2}$ $\bar{3}4\bar{1}2$ $\bar{3}412$ $34\bar{1}2$ $4\bar{1}3\bar{2}$ $413\bar{2}$ $\bar{4}231$ $432\bar{1}$	B and D
$\bar{1}\bar{3}\bar{2}$ $\bar{1}4\bar{3}2$ $\bar{2}1\bar{3}\bar{4}$ $2\bar{1}\bar{3}\bar{4}$ $21\bar{3}\bar{4}$ $\bar{2}\bar{3}1\bar{4}$ $2\bar{3}1\bar{4}$ $\bar{2}\bar{4}3\bar{1}$ $\bar{2}4\bar{3}\bar{1}$ $24\bar{3}\bar{1}$ $2\bar{4}3\bar{1}$ $\bar{2}4\bar{3}1$ $3\bar{1}\bar{2}\bar{4}$ $31\bar{2}\bar{4}$ $3\bar{2}1\bar{4}$ $3\bar{2}\bar{4}1$ $\bar{3}\bar{4}1\bar{2}$ $3\bar{4}\bar{1}\bar{2}$ $3\bar{4}1\bar{2}$ $\bar{3}\bar{4}\bar{2}1$ $34\bar{2}\bar{1}$ $\bar{3}4\bar{2}1$ $3\bar{4}\bar{2}1$ $\bar{4}\bar{1}\bar{3}2$ $\bar{4}\bar{1}3\bar{2}$ $\bar{4}13\bar{2}$ $4\bar{1}\bar{3}2$ $\bar{4}1\bar{3}2$ $4\bar{2}1\bar{3}$ $4\bar{2}\bar{3}\bar{1}$ $\bar{4}\bar{2}\bar{3}1$ $\bar{4}23\bar{1}$ $4\bar{2}\bar{3}1$ $4\bar{3}\bar{1}\bar{2}$ $4\bar{3}\bar{1}2$ $\bar{4}\bar{3}12$ $4\bar{3}1\bar{2}$ $4\bar{3}\bar{2}1$	D

Table I.2: Bad patterns for the three types of groups discussed.

S. Billey and A. Postnikov later generalised the whole pattern avoidance approach by restating these results uniformly in a Lie theoretic way in terms of patterns in root sub-systems with star shaped Coxeter diagrams. They thus attained smoothness results in the sporadic types, and were able to characterise other classes of group elements in terms of pattern avoidance; for details see [3]. For an in-depth survey of results on the smoothness of Schubert varieties up to the year 2000 (which does not include [3], or the results in the rest of this report), consult [2].

II Smoothness and Hyperplane Arrangements

In this chapter we discuss the work of S. Oh, A. Postnikov, and H. Yoo in [17] and [18]. Their result is as follows: fix a Weyl group W with Coxeter system (W, S), and let $w \in W$. Define the **inversion arrangement** of w to be the the sub-hyperplane arrangement of $\mathcal{H}$ whose hyperplanes correspond to the set of inversions of w, $\mathrm{inv}(w) \subset R(W, S)$. We denote this hyperplane arrangement $\mathcal{A}_w$ and consider the regions $V \backslash \mathcal{A}_w$. Let r_0 be the region which contains the fundamental chamber of $\mathcal{H}$, C; then just as before we can define a distance function d of the regions, so that $d(r_0, r)$ is the number of hyperplanes in $\mathcal{A}_w$ which separate the region r from r_0. We then define the polynomial $R_w(q)$ to be the generating function which counts the regions in $\mathcal{A}_w$ according to their distance from r_0, i.e.

$$R_w(q) = \sum_r q^{d(r_0,r)}$$

where the sum is over all the regions of $\mathcal{A}_w$. The theorem is then:

Theorem II.1. *[17, 18] Let $X(w)$ be a Schubert variety with $w \in W$ some Weyl group. $X(w)$ is rationally smooth if and only if $P_w(q) = R_w(q)$.*

The work of V. V. Doedhar and D. Peterson mentioned previously means that this condition in fact characterises smoothness in types A, D and E. Since Weyl groups are finite, $\mathcal{A}_w$ is a central arrangement (i.e. all of the hyperplanes pass through the origin), so $R_w(q)$ is always palindromic. Thus Theorem I.3 immediately gives us that if $X(w)$ is *not* rationally smooth, then $P_w(q) \neq R_w(q)$, so we need only prove one direction. The first section below focuses on type A, and the other types are treated in the second section. In all cases, the key is finding a way to factorise the polynomials and showing that they share the same factors.

II.1 Smoothness in Type A

Recall that $A_{n-1} \cong S_n$, throughout this section elements $w \in S_n$ will be written in one-line notation. There is a straightforward way to characterise the inversions of a permutation written in one-line notation. Note that the reflections in S_n are exactly the transpositions. The transposition which swaps $1 \leq i < j \leq n$ is an inversion of w if and only if $w(i) > w(j)$, or in other words $fl(w(i), w(j)) = 21$.

1A Graphical Hyperplane Arrangements

The inversion arrangement of w turns out to be a special kind of hyperplane arrangement called a graphical arrangement, for a thorough treatment of these, consult [19, Section 2.4].

Definition II.1. Let Γ be a graph with vertex set $\{1, \ldots, n\}$, then the **graphical hyperplane arrangement** of Γ in $\mathbb{R}^n$, $\mathcal{A}_\Gamma$, consists of the hyperplanes $x_i - x_j = 0$ for all edges (i, j) in Γ.

Let Γ_w be the graph with edge (i, j) if and only if the transposition swapping i and j is an inversion of w. Then we claim that the restriction of $\mathcal{A}_{\Gamma_w}$ to the hyperplane orthogonal to the vector $(1, 1, \ldots, 1)$ in $\mathbb{R}^n$ coincides with $\mathcal{A}_w$; in particular the region generating functions of the two arrangements coincide. Indeed this follows from the fact that the hyperplane arrangement $\mathcal{H}$ of A_{n-1} (which lives in $\mathbb{R}^{n-1}$) is exactly the restriction to the hyperplane orthogonal to the vector $(1, 1, \ldots, 1)$ in $\mathbb{R}^n$, of the hyperplane arrangement consisting of all hyperplanes $x_i - x_j = 0$ for $1 \leq i < j \leq n$.

There is a bijection between the regions of the graphical arrangement of a graph Γ and acyclic orientations of Γ (i.e. an orientation such that Γ contains no oriented cycles). A region r is defined by inequalities of the form $x_i < x_j$ for $1 \leq i, j \leq n$, associate to r the orientation $\mathcal{O}$ in which $i \to j$ for each edge (i, j). It can be shown that this orientation is acyclic, and every acyclic orientation gives rise to a region [19, Lemma 2.93]. Moreover the distance $d(r_0, r)$ can also be read off from $\mathcal{O}$.

Definition II.2. Let Γ be a graph on the vertex set $\{1,\dots,n\}$, and $\mathcal{O}$ an acyclic orientation of Γ. Define the **descent number** of $\mathcal{O}$ to be

$$\mathrm{des}(\mathcal{O}) = \#\{i \to j \mid i > j\}$$

i.e. the number of descent edges in $\mathcal{O}$. Now define the polynomial

$$R_\Gamma(q) = \sum_{\mathcal{O}} q^{\mathrm{des}(\mathcal{O})}$$

summed over all acyclic orientations of Γ.

The region r_0 in $\mathcal{A}_w$ is defined by $x_i < x_j$ for all edges (i,j) of Γ_w with $i < j$, so it is clear that if the corresponding acyclic orientation is $\mathcal{O}_0$, then $\mathrm{des}(\mathcal{O}_0) = 0$. Now let r be an arbitrary region, with $d(r_0, r) = d$, so there are d hyperplanes separating r_0 and r. This corresponds to swapping the region-defining inequality $x_i < x_j$ for $x_i > x_j$ when (i,j) an edge of Γ_w with $i < j$. This swaps the orientation of d edges in $\mathcal{O}_0$ to give the acyclic orientation $\mathcal{O}$ corresponding to r, and thus contributing d to $\mathrm{des}(\mathcal{O})$. Hence $\mathrm{des}(\mathcal{O}) = d(r_0, r)$ and

$$R_{\Gamma_w}(q) = R_w(q)$$

for all permutations w.

In order to prove equality with the Poincaré polynomial $P_w(q)$ we will make use of the following result by V. Gasharov.

Theorem II.2. *[11, Theorem 1.1] Let $w \in S_n$, then $P_w(q)$ factors into polynomials of the form $1 + q + \cdots + q^r$ if and only if w avoids the two patterns 4231 and 3412.*

We define the notation $[e+1]_q = \frac{1-q^{e+1}}{1-q} = 1 + q + \cdots + q^e$ for e a non-negative integer, this is the **q-number** corresponding to e. Because of this theorem we want to be able to factor $R_\Gamma(q)$, we can do this inductively using the following result.

Lemma II.1. *[6, Theorem 2.4] Let Γ be a graph on the set $\{1,\dots,n\}$, and suppose there is a vertex v which satisfies:*

1. *The set of neighbours of v forms a complete subgraph of Γ, and*
2. *Either*
   - *(a) all neighbours of v are less than v, or*
   - *(b) all neighbours of v are greater than v*

   in the usual ordering on $\{1,\dots,n\}$.

Then $R_\Gamma(q) = [e+1]_q R_{\Gamma\backslash v}(q)$, where e is the number of neighbours of v, and $\Gamma\backslash v$ is the graph obtained by removing v and all its incident edges from Γ.

Proof. Let $\mathcal{O}$ be a fixed acyclic orientation of $\Gamma\backslash v$, we want to count all the ways of extending it to an acyclic orientation on Γ. The vertex v has the set $N \subset \{1,\dots,n\}$ of e neighbours in $\Gamma\backslash v$ which form a a complete subgraph $\Gamma|_N \simeq K_e$.

There are $e+1$ ways to extend an acyclic orientation on K_e to an acyclic orientation on K_{e+1} which is easily seen by induction. Let $\mathcal{O}$ be the acyclic orientation on K_e, then necessarily K_e has a sink vertex, call this t. If we extend $\mathcal{O}$ to $\mathcal{O}'$ on K_{e+1} such that $t \to v$, then t is no longer a sink, and so v must be a sink, and so $\mathcal{O}'$ is unique. Otherwise t remains a sink in $\mathcal{O}'$. Remove t from K_e and K_{e+1} to get K'_{e-1} and K'_e, and now the question is how many ways are there to extend and acyclic orientation on K'_{e-1} to one on K'_e when v is added. By induction this is e, so in total we have $e+1$ ways to extend $\mathcal{O}$.

In fact this shows that for each $j = 0,\dots,e$ there is a unique extension of $\mathcal{O}$ to an acyclic orientation $\mathcal{O}'$ so that there are exactly j edges oriented towards v. All vertices in N are less than v or greater than v; in either case we have

$$\sum_{\mathcal{O}'} q^{\mathrm{des}(\mathcal{O}')} = [e+1]_q q^{\mathrm{des}(\mathcal{O})}$$

where the sum is over all extensions $\mathcal{O}'$ of $\mathcal{O}$. The term $q^j q^{\mathrm{des}(\mathcal{O})}$ corresponds to the extension of $\mathcal{O}$ with j new descent edges whence the claim. ■

Thus in order to factorise $R_{\Gamma_w}(q)$ we want to show that we can always find a vertex v with the properties in the above lemma. In fact to completely factorise the polynomial, we want to find a sequence containing all of the vertices such that the graph we get by restricting to any truncation of this sequence always has a vertex (the last one in the truncated sequence) with the above properties. We construct such a sequence in the next section.

1B Perfect Elimination Orderings

Definition II.3. A **perfect elimination ordering** of Γ is an ordering of its vertices $v_1 < \cdots < v_n$, such that for any vertex v_i, its neighbours which are less than it define a complete subgraph of Γ. Given such an ordering, let e_i be the number of neighbours of v_i among $v_1, \ldots, v_{i-1}$; the numbers $e_1, \ldots, e_n$ are the **exponents** of Γ, which correspond to e in Lemma II.1.

We say that a perfect elimination ordering of Γ is **nice** if for any vertex v_i, its neighbours which are less than it in the ordering are either all less than, or all greater than v_i in the usual ordering on $\{1, \ldots, n\}$.

It is clear that if $v_1, \ldots, v_n$ is a nice perfect elimination ordering for Γ, v_n satisfies the conditions of Lemma II.1, and moreover $v_1, \ldots, v_{n-1}$ is a nice perfect elimination ordering for $\Gamma \backslash v_n$. Hence the following.

Corollary II.1. *Let Γ be a graph with a nice perfect elimination ordering, whose exponents are $e_1, \ldots, e_n$, then*

$$R_\Gamma(q) = [e_1 + 1]_q \cdots [e_n + 1]_q$$

From this it is clear that if Γ possesses a *nice* perfect elimination ordering, the exponents do not depend on which nice ordering we take. In fact the ordering does not have to be nice in order to calculate the exponents (this can be proved in greater generality, see Lemma III.2).

Lemma II.2. *[17, Proposition 12] The set of exponents of Γ does not depend on the choice of perfect elimination ordering.*

Proof. Let $\chi_\Gamma(t)$ be the *chromatic polynomial* of Γ, i.e. for $m \in \mathbb{N}$, $\chi_\Gamma(m)$ is the number of colourings of the vertices of Γ such that no two neighbouring vertices have the same colour. Let $v_1, \ldots, v_n$ be a perfect elimination ordering of Γ with exponents $e_1, \ldots, e_n$, then we claim

$$\chi_\Gamma(t) = (t - e_1) \cdots (t - e_n)$$

Indeed it is sufficient to prove it for an arbitrary $m \in \mathbb{N}$. We count the number of colourings: the vertex v_1 can be coloured in $m \overset{e_1=0}{=} m - e_1$ colours, vertex v_2 in $m - e_2$ colours, and in general the vertex v_i can be coloured in $m - e_i$ colours since the e_i preceding vertices have already been coloured using e_i different colours.

It is clear that $\chi_\Gamma(t)$ is independent of the choice of ordering, hence so is the set $\{e_1, \ldots, e_n\}$. ■

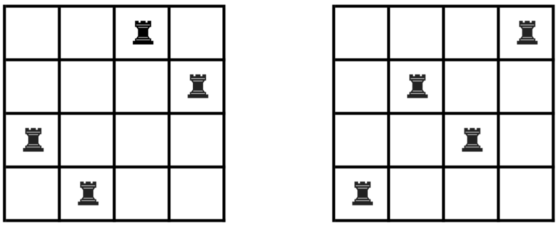
Let Γ_w be the inversion graph for $w \in S_n$, we shall now construct a nice perfect elimination ordering of its vertices. We want to represent w using a **rook diagram** D_w, this is an $n \times n$ chess board on which we place n non-attacking rooks at positions $(w(i), i)$ for each $1 \leq i \leq n$, where (x, y) corresponds to the x^{th} box down and the y^{th} box across. This is the shape of the corresponding permutation matrix. The rook diagrams for the two bad patterns in type A (see Theorem I.4) are shown in Figure II.1a.

The graph Γ_w contains an edge (i, j) with $i < j$ whenever the rook in the i^{th} column of D_w is south-west of the rook in the j^{th} column. Then we say that this pair of rooks **forms an inversion**. Let a be the rook in the n^{th} column of D_w, and b the rook in the n^{th} row. We use a and b to divide up D_w into four (possibly empty) regions A, B, C, and D, see Figure II.1b. If $w(n) = n$ we take $a = b$.

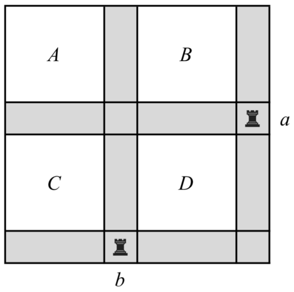
Lemma II.3. *[17, Lemma 17] Let w be a rationally smooth permutation, then its rook diagram D_w has the properties:*

1. *Each pair of rooks in region D forms an inversion, and*
2. *At least one of the sectors B or C contains no rooks.*

If not, D_w would contain a sub-diagram which looks like one of those in Figure II.1a, and so w would contain the corresponding bad pattern.



(a) Rook diagrams representing the permutations 3412 and 4231 in S_4.



(b) Rook diagram decomposition.

Figure II.1

We shall use this to construct a nice perfect elimination ordering of Γ_w recursively when w is rationally smooth. If w is rationally smooth, then D_w satisfies the two properties in Lemma II.3. Let us assume that in property 2, it is region B which is empty, the case C is empty is analogous. Let $v_a = n$ and v_b be the vertices of Γ_w corresponding to rooks a and b. The neighbours of v_b are either the vertices whose rooks are in region D which we shall call $v_1, \ldots, v_k$, or v_a. Hence v_b is less than all of its neighbours in the usual ordering on $\{1, \ldots, n\}$. Moreover, every pair of neighbours of v_b forms an inversion, so the neighbours of v_b form a complete subgraph of Γ_w. This is illustrated by Figure II.2 (taking C empty instead, we would have that v_a is greater than all of its neighbours, which form a complete subgraph).

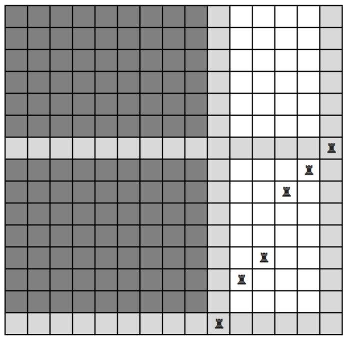


Figure II.2: A possible rook diagram for a rationally smooth permutation w, in which the region B is empty. We are not concerned with the contents of A and C.

Using $\prec$ to denote the ordering we are constructing, this tells us we need to take $v_1, \ldots, v_k, v_a \prec v_b$ (respectively take $v_1, \ldots, v_k, v_b \prec v_a$). Let w' be the permutation $w(1)\cdots w(v_b - 1)w(v_b + 1)\cdots w(n)$, and $D_{w'}$ the corresponding rook diagram (i.e. with the ${v_b}^{\text{th}}$ column and $w(v_b)^{\text{th}}$ row removed from D_w). w' is clearly still a rationally smooth permutation (since it will still avoid the bad patterns), so we can apply the same procedure as above, and thus recursively construct a series of relations among the vertices. Then take any ordering which satisfies these relations (an ordering will always exist), and this will be a nice perfect elimination ordering on Γ_w.

In fact we can also calculate the exponents using Lemma II.1. If region B is empty, then when we remove v_b we get the factor $[e+1]_q$ where $e = n - v_b$ is the number of neighbours of v_b as in Lemma II.1. If region C is empty, then we get the same factor by removing v_a, but with $e = n - v_a$.

1C Factorisation of $R_w(q)$ and the Proof of the Theorem

By showing that Γ_w has a nice perfect elimination ordering when w is smooth, we can factorise the polynomial $R_\Gamma(q)$ using Corollary II.1 as

$$R_\Gamma(q) = [e_1 + 1]_q \cdots [e_n + 1]_q$$

where $e_1, \ldots, e_n$ are the exponents of Γ_w. With this recursively defined ordering, the exponents take a long time to compute. However Lemma II.2 tells us that we can use any perfect elimination ordering to compute the exponents, it does not have to be a nice one. Here we shall give a perfect elimination ordering with which we can easily compute the exponents.

Definition II.4. Let $w \in S_n$ be a permutation, $r \in \{1, \ldots, n\}$ is a **record position** for w if

$$w(r) > \max\{w(1), \ldots, w(r-1)\}.$$

In the one-line notation, the record positions are the positions of the left-to-right maxima. Then we call $w(r)$ a **record**.

Lemma II.4. *[17, Lemma 22] Let $w \in S_n$ be a rationally smooth permutation with record positions $r_1 = 1 < r_2 < \cdots < r_s$, then the ordering*

$$[r_s, n], [r_{s-1}, r_s - 1], \ldots, [r_2, r_3 - 1], [r_1, r_2 - 1]$$

is a perfect elimination ordering of Γ_w, where $[a, b]$ denotes $\{a, a+1, \ldots, b-1, b\}$ with the usual ordering.

The proof is not hard, but quite technical so we omit it. Given this ordering, the exponents can readily be calculated as follows. For $i \in \{1, \ldots, n\}$, let r and r' be the record positions of w such that $r \leq i < r'$, and there are no record positions between r and r' (let $r' = +\infty$ if there are no record positions greater than i). Then

$$e_i = \#\{j \mid r \leq j < i, w(j) > w(i)\} + \#\{k \mid r' \leq k \leq n, w(k) < w(i)\} \tag{II.1}$$

Indeed, $i \in [r, r'-1]$; an index j contributes to e_i if $w(j) < w(i)$ (so the vertex j is a neighbour of i), and $j < i$ in the usual ordering. If $j \in [r, r'-1]$ it contributes to the first term, otherwise it must be in $[r', r''-1], \ldots, [r_2, r_3 - 1], [r_1, r_2 - 1]$, in which case it contributes to the second term (r'' being the next record position after r'). We can now prove the main theorem of this section.

Theorem II.3. *[17, Theorem 7] Let $X(w)$ be a Schubert variety with $w \in S_n$. $X(w)$ is rationally smooth if and only if $P_w(q) = R_w(q)$.*

Proof. As discussed at the start of this chapter, we need only prove that if w is smooth, then we have the equality $P_w(q) = R_w(q)$. Let w be smooth, and let $\mathscr{A}_w$ be the corresponding inversion hyperplane arrangement, then $R_w(q)$ is the region generating function of $\mathscr{A}_w$. This is in fact a graphical hyperplane arrangement, so let Γ_w be the inversion graph of w. Then we have the equality $R_w(q) = R_{\Gamma_w}(q)$ where the second is the generating function of the acyclic orientations on Γ_w. We have constructed a nice perfect elimination ordering of Γ_w, so by Corollary II.1, $R_{\Gamma_w}(q)$ factors as a product of factors $[e_i + 1]_q$, where the the exponents can be calculated using (II.1).

Finally we use the result of V. Gasharov (Theorem II.2) which says that $P_w(q)$ also factorises, with factors of the same form. At the end of the paper ([11, Remark 2.8]), V. Gasharov remarks that V. Reiner pointed out that the exponents in $P_w(q)$ are exactly those which we calculated recursively (i.e. $n - v_a$ or $n - v_b$ in the two cases). Hence $P_w(q) = R_w(q)$ for w rationally smooth. ∎

Remark II.1. Note that we have not constructed an explicit bijection between the regions of $\mathscr{A}_w$ and elements of the Bruhat ideal $[e, w]$. Indeed we shall see in general that the regions of $\mathscr{A}_w$ do not correspond geometrically to the chambers in the hyperplane arrangement $\mathcal{H}$ which are labelled by the elements in $[e, w]$ (see Example III.2).

Example II.1. Let us illustrate the main ideas in the proof with an example. Let $w = 65174832 \in S_8$ which we can see by observation contains neither of the patterns 3421 or 4231, so $X(w)$ is rationally smooth. Either by applying Gasharov's theorem, or as we did, direct computation using a computer [20, Section 8], we find that the Poincaré polynomial of w is

$$P_w(q) = [0+1]_q[1+1]_q[2+1]_q^3[3+1]_q^2[4+1]_q$$

so the set of exponents of w should be $\{0, 1, 2, 2, 2, 3, 3, 4\}$.

We shall now recursively compute a nice perfect elimination ordering of $\{1, \ldots, 8\}$ for w. Figure II.3 shows the rook diagrams at each stage with arrows indicating the rook we remove; the order relations we get are as follows:

① $7, 8 \prec 6$ ③ $1, 2, 5, 7 \prec 8$ ⑤ $1, 2 \prec 5$ ⑦ $3 \prec 2$

② $5, 7, 8 \prec 4$ ④ $1, 2, 5 \prec 7$ ⑥ $2, 3 \prec 1$

These relations force the relative ordering

$$3 \prec 2 \prec 1 \prec 5 \prec 7 \prec 8 \prec 4, 6$$

but we have free choice of the order of 4 and 6; either order will give a nice perfect elimination ordering. We shall take $4 \prec 6$.

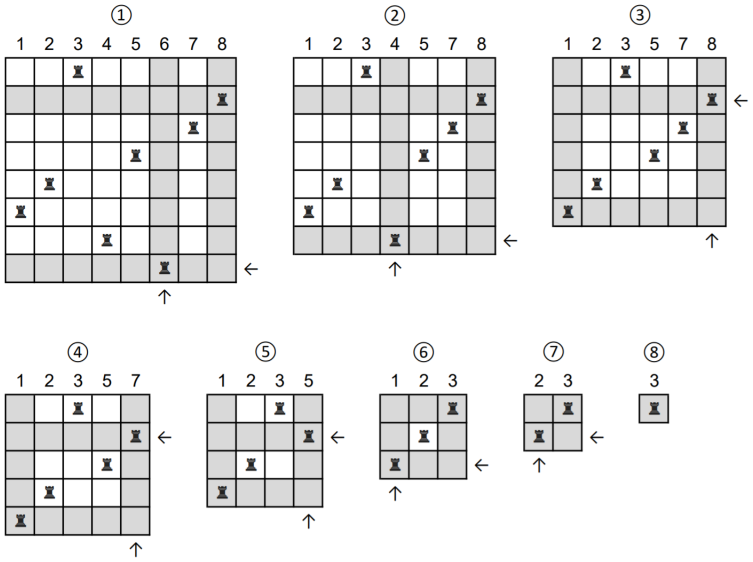


Figure II.3: The rook diagrams for recursively computing a nice perfect elimination ordering for w.

We can now draw the inversion graph Γ_w with this ordering in mind, and quickly check visually that indeed given any vertex, all of its neighbours which are less than it form a complete subgraph, and they are either all greater, or all less than the given vertex in the usual ordering, see Figure II.4.

Finally we want to calculate the exponents, and hence factor $R_w(q)$. We shall use the simple perfect elimination based on the record positions. The record positions for $w = 65174832$ are 1, 4, and 6, with records 6, 7 and 8 respectively. Hence a perfect elimination ordering is given by

$$\underbrace{6, 7, 8}, \underbrace{4, 5}, \underbrace{1, 2, 3}$$

from which we can calculate the exponents using (II.1).

$$e_1 = 0 + 3 = 3 \qquad e_2 = 1 + 3 = 4 \qquad e_3 = 2 + 0 = 2 \qquad e_4 = 0 + 2 = 2$$
$$e_5 = 1 + 2 = 3 \qquad e_6 = 0 + 0 = 0 \qquad e_7 = 1 + 0 = 1 \qquad e_8 = 2 + 0 = 2$$

This gives the same set of exponents $\{0, 1, 2, 2, 2, 3, 3, 4\}$ and factorisation as with $P_w(q)$

$$\begin{aligned} R_w(q) &= [0+1]_q[1+1]_q[2+1]_q^3[3+1]_q^2[4+1]_q \\ &= 1 + 7q + 27q^2 + 74q^3 + 159q^4 + 282q^5 + 425q^6 + 554q^7 + 631q^8 \\ &\quad + 631q^9 + 554q^{10} + 425q^{11} + 282q^{12} + 159q^{13} + 74q^{14} + 27q^{15} + 7q^{16} + q^{17} \end{aligned}$$

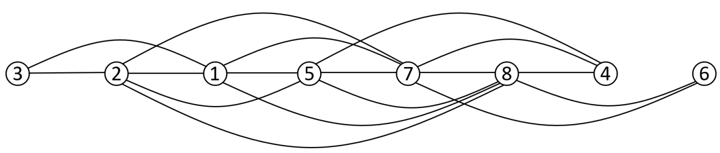


Figure II.4: The inversion graph for w, with the vertices arrange in the nice perfect elimination ordering constructed above.

II.2 Smoothness in other Types

The proofs for types $B = C$ and D follow the same outline as the one for type A, and rely on the combinatorial description of these groups which we introduced in Section I.3B. We define an inversion hyperplane arrangement associated to and element w, and define $R_w(q)$ to be the region counting polynomial. We show the equivalence of this polynomial with the polynomial counting certain types of acyclic orientations on the inversion graph of w, which is analogous to Γ_w for type A. We then factorise $R_w(q)$ and use results analogous to the theorem of V. Gasharov which were proved by S. Billey in [1]. Because of these similarities, we shall just sketch the relevant definitions and results, for details the interested reader should consult [18], or H. Yoo's thesis [24].

The remaining types are E, F, and G. G_2 is simple, we shall prove the result uniformly for all dihedral groups in the next chapter, see Example III.2. The authors of [18] claim to have checked F_4 with a computer. For type E the authors show that these cases can be proved uniformly with type A and D, once a certain decomposition property of elements in A and D is shown to hold true for type E as well; again they did this using a computer, see [18, Propositions 8 and 13]. They do not give details of these computer checks, so we shall forego saying anything else about these cases.

2A Type B

We shall follow the notation and exposition in [24]. Let S_n^B be the Weyl group of type B and rank n, and recall that it is the group of signed permutations. In particular it is the subgroup of the permutations of $[\pm n] := \{-n, \dots, -1, 1, \dots, n\}$ such that $w(-a) = -w(a)$ for all $w \in S_n^B$ (we also define the notation $[n] := \{1, \dots, n\}$). Table I.2 summarises the patterns which w avoids if and only if it is rationally smooth. We can characterise the set $\operatorname{inv}(w)$ combinatorially, as with elements of S_n.

$$\operatorname{inv}(w) = \{(i,j) \in [n] \times [n] \mid i < j,\ w(i) > w(j)\} \cup \{(i,j) \in [n] \times [n] \mid i < j,\ w(-i) > w(j)\} \qquad \text{(II.2)}$$

Definition II.5. Let $w \in S_n^B$, the **inversion graph** of w, Γ_w^B has vertex set $[\pm n]$, with single edges $\{(i,j) \in [\pm n] \times [\pm n] \mid i < j,\ i \neq -j,\ w(i) > w(j)\}$, and double edges $\{(-i,i) \mid i \in [n],\ w(-i) > w(i)\}$.

The reason we have double edges, is because each single edge (i,j) is paired with another edge $(-j,-i)$ in Γ_w^B, so we shall always consider pairs of edges. We use these pairs to define the inversion arrangement associated to w.

Definition II.6. Let $w \in S_n^B$, the **inversion hyperplane arrangement** $\mathcal{A}_w^B$ associated to W is the hyperplane arrangement in $\mathbb{R}^n$ with hyperplanes:

- $x_i - x_j = 0$ for all pairs of edges $\{(-j,-i),(i,j)\}$ in Γ_w^B where $0 < i < j$, and
- $x_{-i} + x_j = 0$ for all pairs of edges $\{(-j,-i),(i,j)\}$ in Γ_w^B where $i < 0 < j$.

Definition II.7. An orientation on Γ_w^B is **asymmetric** if the direction of (i,j) and $(-j,-i)$ are the same, then the direction of the pair $\{(-j,-i),(i,j)\}$ determines which half-space of the corresponding hyperplane a point is in. Then just as in the case of S_n, the regions of $\mathcal{A}_w^B$ are in bijection with the acyclic antisymmetric orientations on Γ_w^B.

As before, $\mathcal{A}_w^B$ is a sub-hyperplane arrangement of the hyperplane arrangement $\mathcal{H}$ associated to B_n. Let r_0 be the region of $\mathcal{A}_w^B$ induced by the fundamental chamber in $\mathcal{H}$. Then we define

$$R_w(q) = \sum_r q^{d(r_0,r)}$$

summed over all regions of $\mathcal{A}_w^B$.

Definition II.8. Let $\mathcal{O}$ be an acyclic asymmetric orientation of Γ_w^B, and define $\operatorname{des}^B(\mathcal{O})$ to be the number of pairs $\{(-j,-i),(i,j)\}$ oriented as $-j \to -i$ and $i \to j$ in $\mathcal{O}$, where $i > j$, i.e. the number of descent edge pairs.

As before we have the equality

$$R_w(q) = \sum_{\mathcal{O}} q^{\operatorname{des}^B(\mathcal{O})}$$

where the sum is over all acyclic antisymmetric orientations of $\mathcal{A}_w^B$.

The last big piece of the puzzle is the factorisation of $P_w(q)$, analogous to Gasharov's theorem. Recall the notation introduced for the generators of S_n^B in Section I.I.3.

Theorem II.4. *[1, Theorem 3.3] Let $w \in S_n^B$ and assume $w(d) = \pm n$ and $w(n) = \pm e$. Then $P_w(q)$ factors in the form*

$$P_w(q) = P_{w'}(q)[\mu + 1]_q$$

under the following circumstances:

1. *If $w(d) = n$ and $w(d) > w(d+1) > \cdots > w(n)$, then $w' = w\sigma_d^B \cdots \sigma_{n-1}^B$, and $\mu = n - d$.*
2. *If w^{-1} is in the previous situation, then $w' = \sigma_{n-1}^B \cdots \sigma_{e+1}^B \sigma_e^B w$ and $\mu = n - e$.*
3. *If each $w(i)$ is negative and $w(1) > w(2) > \cdots > w(d-1) > w(d+1) > \cdots > w(n)$, then $w' = w\sigma_{d-1}^B \cdots \sigma_1^B \sigma_\pm^B \sigma_1^B \cdots \sigma_{n-1}^B$ and $\mu = d + n - 1$.*
4. *If w^{-1} is in the previous situation, then $w' = \sigma_{n-1}^B \cdots \sigma_1^B \sigma_\pm^B \sigma_1^B \cdots \sigma_{e-1}^B w$ and $\mu = e + n - 1$.*
5. *If each $w(i)$ is positive except for $w(d) = \bar{n}$ and $w(1) > w(2) > \cdots > w(d)$, then $w' = w\sigma_{d-1}^B \cdots \sigma_1^B \sigma_\pm^B$ and $\mu = d$.*

Moreover, if w is rationally smooth then it falls into one of the above circumstances, so $P_w(q)$ factors into q-numbers.

S. Oh and H. Yoo were able to factorise $R_w(q)$ by a careful analysis of each of the above cases, for details, see [24, Section 1.3.5].

2B Type D

We write S_n^D for the Weyl group of type D and rank n. For $w \in S_n^D$, the set of inversions has exactly the same characterisation as in type B, see (II.2). The inversion graph Γ_w^D only has the single edges $\{(i,j) \in [\pm n] \times [\pm n] \mid i < j,\ i \neq -j,\ w(i) > w(j)\}$, but these still come in pairs, and so the inversion hyperplane arrangement is defined in the same way. Again we consider acyclic asymmetric orientations of Γ_w^D, and can conclude that

$$R_w(q) = \sum_{\circledcirc} q^{\mathrm{des}^D(\circledcirc)}$$

where $\mathrm{des}^D(\circledcirc)$ has the same definition as for type B. The factorisation of $P_w(q)$ is again due to S. Billey.

Theorem II.5. *[1, Theorem 6.3] Let $w \in S_n^D$ and assume $w(d) = \pm n$ and $w(n) = \pm e$. Then $P_w(q)$ factors in the form*

$$P_w(q) = P_{w'}(q)[\mu + 1]_q$$

under the following circumstances:

1. *if $w = w_0$ is the longest element in S_n^D, i.e. $w = \pm 1\bar{2} \cdots \bar{n}$, then*

$$P_{w_0}(q) = \prod_{k=1}^{n-1} (1 + q + \cdots + q^{k-1} + 2q^k + q^{k+1} + \cdots + q^{2k})$$

2. *If $w(d) = n$ and $w(d) > w(d+1) > \cdots > w(n)$, then $w' = w\sigma_d^D \cdots \sigma_{n-1}^D$, and $\mu = n - d$.*
3. *If w^{-1} is in the previous situation, then $w' = \sigma_{n-1}^D \cdots \sigma_{e+1}^D \sigma_e^D w$ and $\mu = n - e$.*
4. *If $w(1) < 0$ and $w(d) = \bar{n}$ are the only two negatives and $-w(1) > w(2) > \cdots > w(d)$, then $w' = w\sigma_{d-1}^D \cdots \sigma_2^D \sigma_\pm^D$ and $\mu = d - 1$.*
5. *If $\bar{n}$ and $\bar{1}$ are the only two negatives in the one-line notation and $w(1) > \cdots > w(d)$, then $w' = w\sigma_{d-1}^D \cdots \sigma_2^D \sigma_1^D$ and $\mu = d - 1$.*

Moreover, if w is rationally smooth then it falls into one of the above circumstances. Since it is well-known that $P_{w_0}(q)$ factors into q-numbers (see Theorem III.1), $P_w(q)$ factors into q-numbers for all rationally smooth w.

Again the factorisation of $R_w(q)$ is done by a careful analysis of these cases.

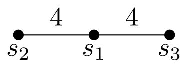
III Generalisation to other Coxeter Groups

At the end of [18] the authors make the following conjecture:

Conjecture. *Let W be any Coxeter group. Then $[e, w]$ is palindromic if and only if $P_w(q) = R_w(q)$.*

In this generality we no longer have Schubert varieties, nevertheless we shall call $w \in W$ **smooth** if $[e, w]$ is palindromic (i.e. $P_w(q)$ is palindromic). It is worth noting that there are flag manifolds and Schubert varieties associated to so-called affine Weyl groups, which are Euclidean Coxeter groups and hence infinite. Theorem I.3 holds in these cases, and work of S. Billey and A. Crites [4] generalised the pattern avoidance in type A. The conjecture fails even for many small examples in infinite Coxeter groups.

Example III.1. Let $W = \tilde{B}_2$ be the affine Coxeter group with Coxeter diagram



which is the symmetry group of the tiling of the plane by squares. Let $w = s_1 s_2 s_3$, using the geometric interpretation both of the Bruhat ideal $[e, w]$ and the inversion set $\mathrm{inv}(w)$ we can represent these as in Figure III.1. From this we can easily read off that

$$P_w(q) = 1 + 3q + 3q^2 + q^3$$

which is palindromic, however

$$R_w(q) = 1 + 3q + 2q^2 + q^3$$

which not only does not equal $P_w(q)$, but is not even palindromic as was always the case for the Weyl groups discussed previously.

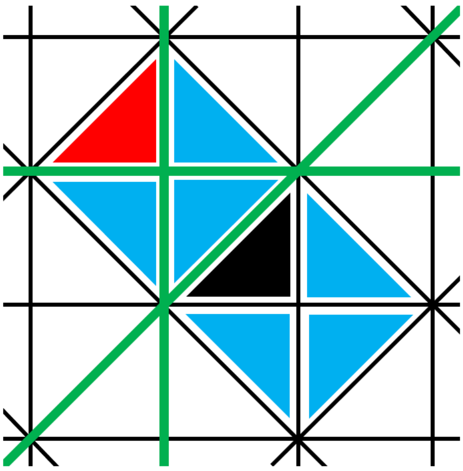


Figure III.1: The Bruhat ideal and inversion arrangement of w. The fundamental chamber is black, the chamber corresponding to w is red, and the other chambers in $[e, w]$ are coloured blue. The inversion arrangement for w is highlighted in green.

In this chapter we shall be exploring the cases (principally when W is finite) in which the conjecture does hold; and in the cases when it fails, studying the Poincaré polynomial geometrically to find other geometric quantities which characterise smoothness, in the place of $R_w(q)$. A lot of the contents is based on the computation of examples, and so this chapter is more speculative than the previous ones.

III.1 Finite Coxeter Groups

Let us summarise what we have seen so far. The finite irreducible Coxeter groups have been classified, and these are listed in Table I.1. Every finite Coxeter group is a Cartesian product of copies of these groups. The classification defines eight types, six types of Weyl groups to which we can associate flag manifolds and Schubert varieties: A, $B = C$, D, E, F, G; along with types H and I which do not have an associated crystallographic root system. The previous chapter was concerned with proving the conjecture for the irreducible Weyl groups, to what extend does it extend to all finite Coxeter groups? In the counter example above we saw that the number of regions in the inversion arrangement was not equal to $\#[e, w]$. The cases when we do have equality for finite Coxeter groups was characterised by A. Hultman in [13, Theorem 3.2].

1A Factorising $P_w(q)$

Before looking at types H and I, we shall state a lemma which is straightforward, but which is nevertheless the key behind the factorisation of the Poincaré polynomial in all cases.

Lemma III.1. *Let (W, S) be a fixed Coxeter system. For a subset Y of W, write $Y(q) = \sum_{w \in Y} q^{l(w)}$ for the length generating function of Y (so $[e, w](q) = P_w(q)$ is the usual Poincaré polynomial). Suppose for a given subset $Y \subset W$, there exist subsets $U, V \subset W$ such that for any $w \in Y$, there are unique elements $u \in U$ and $v \in V$ such that $w = uv$ and $l(w) = l(u) + l(v)$. Moreover assume U and V have the property that $uv \in Y$ for any $u \in U$ and $v \in V$. Then*

$$Y(q) = U(q)V(q)$$

Our first application of this is to consider how one generalises the conjecture from irreducible Coxeter groups to reducible Coxeter groups. We shall say that (W, S) has the **q-factorisation property** if the following holds:

For all $w \in W$, if $P_w(q)$ is palindromic, then it factors completely into a product of q-numbers.

We have seen that if W has type A, $B = C$ or D, then it has the q-factorisation property, and we shall see that types G and I also have this property. We have also confirmed that H_3 satisfies the property using a computer, see [20, Section 12.3].

Proposition III.1. *Let (W, S) be a Coxeter system, with irreducible factors $W_1, \ldots, W_k$. Assume the conjecture holds for all of the irreducible factors.*

1. *If $w_1 \in W_1, \ldots, w_k \in W_k$ is a sequence of elements such that either they are all smooth, or an odd number are non-smooth, and the rest are smooth; then the conjecture holds for $w_1 \cdots w_k \in W$.*
2. *If W has the q-factorisation property, then the conjecture holds.*

Proof. We shall not actually use the fact that the factors are irreducible, so it is sufficient to show that if the conjecture holds for Coxeter systems (W, S) and (W', S), then it holds for $(W \times W', S \cup S')$. Let $w \in W \times W'$. Since the sets S and S' commute in $W \times W'$ we can find a reduced word for w of the form $w = t_1 \cdots t_r t'_1 \cdots t'_s$ where $u = t_1 \cdots t_r \in W$ and $v = t'_1 \cdots t'_s \in W'$. By the subword property of Bruhat order (Theorem I.2), it is clear that the sets $Y = [e, w]$, $U = [e, u]$ and $V = [e, v]$ satisfy the conditions of Lemma III.1, and hence

$$P_w(q) = P_u(q)P_v(q)$$

On the other had, if $\mathcal{H}$ is the hyperplane arrangement associated to W and $\mathcal{H}'$ is the hyperplane arrangement associated to W', then the hyperplane arrangement of $W \times W'$ is the product space $\mathcal{H} \times \mathcal{H}'$. It is a straightforward geometric exercise to see then that

$$R_w(q) = R_u(q)R_v(q)$$

The only impediment to the conjecture holding for $w \in W \times W'$ is if $P_u(q)$ and $P_v(q)$ are not palindromic, but their product is palindromic. In this case w is smooth but we may not have that $P_w(q) = R_w(q)$. This problem is precluded by the hypotheses of (1). In case (2), $P_w(q)$ being palindromic means it factors into q-numbers, each of which factorises into a product of irreducible cyclotomic polynomials, which it is easily seen are always palindromic, so again the problem cannot arise[1]. ■

[1] Our thanks to the first year LSGNT number theorist for this observation.

Question 1. *Do all Coxeter groups have the q-factorisation property?*

In type A we were able to show that for smooth elements, the factorisation into q-numbers was unique. This holds in general.

Lemma III.2. *Let $P(q)$ be a polynomial which factors completely into q-numbers as $P(q) = \prod_{i=1}^{k} [a_i]_q$, then the (multi)set $\{a_1, \ldots, a_k\}$ is unique.*

Proof. Suppose $P(q)$ has another factorisation $P(q) = \prod_{i=1}^{l} [a'_i]_q$. We can write

$$P(q) = \prod_{i=1}^{k} \prod_{1<d|a_i} \Phi_d(q) = \prod_{i=1}^{l} \prod_{1<d'|a'_i} \Phi_{d'}(q)$$

where $\Phi_d(q)$ is the d^{th} cyclotomic polynomial which is irreducible. Suppose a_i is a maximal element with respect to divisibility in $\{a_i\}_{i=1}^{k}$, then it is clear that there must be some index j such that $a'_j = a_i$. Repeating this argument with $P(q)/[a_i]_q$ we see that $k = l$ and $\{a_i\}_{i=1}^{k} = \{a'_j\}_{j=i}^{l}$ as (multi)sets[2]. ■

Recall that for a finite Coxeter group, we denote the unique longest element by w_0, and that $[e, w_0] = W$. C. Chevalley was interested in enumerating Weyl groups by their length, and was able to factorise $W(q) = P_{w_0}(q)$. This was generalised to all finite Coxeter groups by L. Solomon.

Theorem III.1. *[14, Theorem 3.15] Let (W, S) be a finite irreducible Coxeter system of rank n, then there are positive integers $e_1, \ldots, e_n$ such that $P_{w_0}(q) = \prod_{i=1}^{n} [e_i + 1]_q$.*

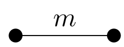
The exponents $e_1, \ldots, e_n$ have been calculated for all finite irreducible Coxeter groups, see [5, Appendix A1]. Since w_0 is always smooth (indeed $w \mapsto w_0 w$ is an anti-automorphism of Bruhat order, see [5, Proposition 2.3.4]), we can generalise this factorisation to all finite Coxeter groups using Proposition III.1; the set of exponents of a reducible finite Coxeter group W is the union of the sets of exponents for its irreducible factors. We shall not give a proof of this theorem, however we shall state an important decomposition result used in the proof. We shall use a variant of this result to factorise Poincaré polynomials for arbitrary W later.

Proposition III.2 (Parabolic Decomposition I)**.** *[14, Proposition 1.10c] Let (W, S) be a Coxeter system. Recall for any subset $T \subset S$, we write $W_T = \langle T \rangle$, and define $W^T := \{w \in W \mid l(sw) > l(w) \text{ for all } s \in T\}$. Given any $w \in W$, for any $T \subset S$ there is a unique $u \in W^T$ and $v \in W_T$ such that $w = uv$ and $l(w) = l(u) + l(v)$. Moreover u is the unique element of smallest length in the coset wW_T.*

1B Types H and I

Proposition III.1 motivates us to look at the remaining irreducible finite Coxeter groups, types H and I.

Example III.2. Consider the Coxeter group W with Coxeter diagram



for $m \in \{2, 3, \ldots, \infty\}$, which includes all Coxeter groups of type I, along with $A_1 \times A_1$, A_2, B_2, G_2, and the irreducible Euclidean Coxeter group $\tilde{A}_1$ (when $m = \infty$), that is all finite dihedral groups together with the infinite dihedral group. Label the two generators a and b, then every element of W has a reduced word which is a finite alternating sequence $\alpha_k = \underbrace{ab\cdots}_{\text{length } k}$ or $\beta_k = \underbrace{ba\cdots}_{\text{length } k}$ for $k \leq m$.

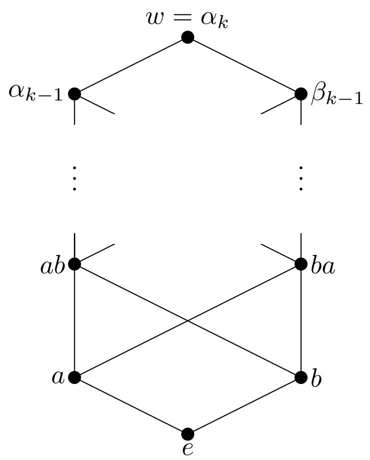
Without loss of generality, let us consider $w = \alpha_k$. The Bruhat covering graph of w is shown in Figure III.2a. It is clear that $[e, w]$ is palindromic, and the Poincaré polynomial of w is

$$P_w(q) = 1 + 2q + 2q^2 + \cdots + 2q^{k-1} + q^k = [2]_q[k]_q$$

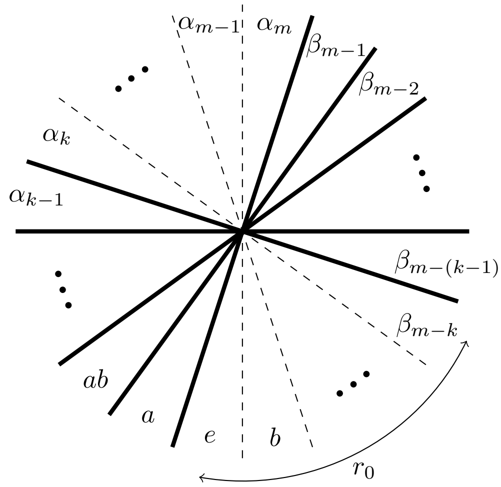
so the exponents are 1 and $k-1$. The inversion arrangement is also easily worked out (see Figure III.2b), and one can see that $R_w(q)$ agrees with $P_w(q)$. Hence every element is smooth, and the conjecture holds for W. Note that Figure III.2 shows that the regions in the inversion arrangement do not correspond to the chambers labelled by the elements of the Bruhat ideal.

Question 2. *Does the conjecture hold in type H?*

[2]The idea of this proof is due to L. La Porta.



(a) The Bruhat covering graph of w.



(b) The inversion arrangement of w for m finite, as a sub-arrangement of the arrangement $\mathcal{H}$ for W. For $m = \infty$ the picture is analogous.

Figure III.2

III.2 Infinite Coxeter Groups

We have seen that the conjecture does not hold when we move to infinite groups. For finite groups $R_w(q)$ is always palindromic because the inversion arrangement is invariant under the antipodal map which behaved well with respect to the length function. For infinite Coxeter groups, there will necessarily be hyperplanes in $\mathcal{H}$ which do not intersect, and so no such antipodal map exists. In this section we shall be looking at the Bruhat ideal $[e, w]$, and the Poincaré polynomial from the point of view of the geometry of $\mathcal{H}$ in order to try to understand when they are palindromic.

2A Automorphisms, Anti-automorphisms, and Isometries of $[e, w]$

As was remarked above, for a finite Coxeter system the map $w \mapsto w_0 w$ is an anti-automorphism of $[e, w_0]$ (i.e. a bijection onto itself which reverses Bruhat order). In general if $[e, w]$ possesses an anti-automorphism then $P_w(q)$ is palindromic, however *a priori* this is not necessary.

Question 3. *Is it the case that $[e, w]$ is palindromic if and only if it possesses an anti-automorphism? Can the existence of an anti-automorphism be characterised, and the anti-automorphism given explicitly?*

On the other hand we might consider the automorphisms of $[e, w]$. The automorphisms of the whole group have been studied and we have the following result.

Theorem III.2. *[12, 23] Let (W, S) be an irreducible Coxeter system of rank at least 3, then the automorphism group of Bruhat order is generated by Coxeter diagram automorphisms and the mapping $w \mapsto w^{-1}$.*

The case of rank 1 is trivial, and rank 2 is exceptional (see [5, Exercise 2.2]), but by Example III.2 we have already solved the conjecture in this case. Algebraically an automorphism of the Coxeter diagram corresponds to a permutation of S which preserves the group relations, hence it extends to an automorphism of the whole group in the obvious way. Geometrically, S corresponds to the faces of the fundamental chamber C (recall from Section I.1C), and the relations correspond to the dihedral angles between these faces. Hence the diagram automorphisms are precisely the symmetries of C, which clearly extend to symmetries of $\mathcal{H}$.

The element w is the unique element of maximal length in $[e, w]$, and since automorphisms of Bruhat order are length preserving, an automorphism α restricts to an automorphism of $[e, w]$ if and only if $\alpha(w) = w$. Geometrically this means a symmetry of C which also stabilises wC if α is a diagram automorphism. Conversely, any automorphism of $[e, w]$ must stabilise C, and hence is an automorphism of Bruhat order arising as in Theorem III.2. Diagram automorphisms fall into a larger class of symmetries of $[e, w]$.

Definition III.1. Let (W, S) be a Coxeter system and $w \in W$. An **isometry** of $[e, w]$ is an isometry of $V \supset \mathcal{H}$ which stabilises $[e, w]\overline{C} = \bigcup_{u \in [e,w]} u\overline{C}$. We write $\mathrm{Isom}([e, w])$ for the group of isometries of $[e, w]$.

It is clear that an isometry of $[e, w]$ must stabilise $\mathcal{H}$ and so is in the group generated by W itself together with the symmetries of C. The following result concerns those elements of W which are isometries of $[e, w]$.

Lemma III.3. *Let (W, S) be a Coxeter system and $w \in W$. Recall the notation $D(w) := S \cap \mathrm{inv}(w)$. If $w' \in W_{D(w)}$ then w' stabilises $[e, w]$.*

Proof. Let $u \in [e, w]$. It is sufficient to show the claim for $s \in D(w)$. If $s \in \mathrm{inv}(u)$, $l(su) < l(u)$ by Proposition I.1 and so $su \xrightarrow{s} u$, in particular $su \leq u \leq w$, so $su \in [e, w]$. Otherwise $s \notin \mathrm{inv}(u)$, so there is no reduced word representing u which starts with s (see Remark I.2). On the other hand there is a reduced word for w starting with s, and so by the subword property, su is a subword of w, hence $su \leq w$. ■

Conjecture 1. *Let (W, S) be a Coxeter system and $w \in W$, then* $\mathrm{Isom}([e, w])$ *is generated by $D(w)$ together with diagram automorphisms which fix w.*

Remark III.1. Evidence to support this conjecture can be seen in Figure III.5 and [20, Section 11.5.3].

Fix $w \in W$, we consider the subgroup $W_{D(w)}$ of $\mathrm{Isom}([e, w])$. Since $[e, w]$ is a finite set (see Remark I.1), $\mathrm{Isom}([e, w])$, and hence $W_{D(w)}$ are finite groups. In fact this second group is a finite Coxeter group, and so we can consider the sub-hyperplane arrangement associated to $W_{D(w)}$ of the hyperplane arrangement $\mathcal{H}$ arising from W, which we shall denote $\mathcal{H}_w$. If C is the fundamental chamber in $\mathcal{H}$ (i.e. the chamber associated to e_W), let C_w be the chamber of $\mathcal{H}_w$ which contains C and thus is associated to $e_{W_{D(w)}}$. Define $\mathcal{F}_w := \{u \in W \mid uC \subset C_w\}$ to be those elements of W whose corresponding chambers lie in C_w. Then $W_{D(w)}$ acts simply transitively on the chambers of $\mathcal{H}_w$ with C_w as fundamental domain. Thus

Lemma III.4 (Parabolic Decomposition II)**.** *Let (W, S) be a Coxeter system, and $w \in W$. Then there exist unique $u \in W_{D(w)}$ and $v \in \mathcal{F}_w$ such that $w = uv$. Moreover $l(w) = l(u) + l(v)$. Since $W_{D(w)} \subset \mathrm{Isom}([e, w])$, $W_{D(w)} \subset [e, w]$, and for any $u \in W_{D(w)}$, and $v \in \mathcal{F}_w \cap [e, w]$, $uv \in [e, w]$. It thus follows from Lemma III.1 that*

$$P_w(q) = P_{w_0}(q)(\mathcal{F}_w \cap [e, w])(q)$$

where w_0 is the longest element in $W_{D(w)}$, and $(\mathcal{F}_T \cap [e, w])(q)$ is the length generating polynomial of the set of elements $\mathcal{F}_T \cap [e, w]$.

Theorem III.1 means that we know $P_{w_0}(q)$, factors into q-numbers, and the exponents have been tabulated. Hence we are reduced to considering just the set $\mathcal{F}_T \cap [e, w]$. Whether or not w is smooth, as well as the full factorisation of $P_w(q)$ is determined by this set.

Remark III.2. Diagram automorphisms are length preserving so they do not contribute to the factorisation pf $P_w(q)$ which is why we do not include these. Algebraically the set $\mathcal{F}_w$ is the same as $W^{D(w)}$, so the content of this lemma is the second bullet point. In fact the set $\mathcal{F}_w$ also admits the description as the set of minimal length coset representatives for $W_{D(w)} \backslash W$. In [12], A. Hombergh showed that for any $w \in W$ and any subset $T \subset S$, W_T contains a *unique* element of maximal length in $[e, w]$, which we denote $m(w, T)$. S. Billey and A. Postnikov proved the following using this.

Lemma III.5 (Parabolic Decomposition III)**.** *[3, Theorem 6.4] Let (W, S) be a Coxeter system, and $w \in W$. Let $T \subset S$ and suppose we have a parabolic decomposition $w = uv$ with $u \in W_T$ and $v \in W^T$, and furthermore, that $u = m(w, T)$. Then*

$$P_w(q) = P_u(q)(W^T \cap [e, v])(q)$$

Remark III.3. If we take $T = D(w)$ in the previous lemma we recover the Lemma III.4, and $T \subset D(w)$ we get a coarser factorisation. If we assume Conjecture 1 and we take $T \subset S \backslash D(w)$ then the factorisation we get is trivial, and taking any other T, the condition $u = m(w, T)$ is not satisfied. Hence the two previous lemmas have the same content, although the geometric approach is, as far as we know, novel. This result was further studied by W. Slofstra [21].

Example III.3. Consider again the symmetry group of the tiling of the plane by squares, $\tilde{B}_2$, and let $w = 121231$ which has Poincaré polynomial $P_w(q) = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6$. We can visualise the Bruhat ideal $[e, w]$ as in Figure III.3. The fundamental chamber C is black and the chamber corresponding to w is red. The sub-hyperplane arrangement $\mathcal{H}_w$ is shown in green. The set of elements $\mathcal{F}_w \cap [e, w]$ corresponds to the two brown chambers. It is clear that the brown chambers together with C tile $[e, w]$. Since the length generating function of these elements is $1 + q + q^2$ which is palindromic, w is smooth, and in fact

$$P_w(q) = P_{w_0}(q)(1 + q + q^2) = (1 + q)(1 + q + q^3)(1 + q + q^2) = [2]_q[3]_q[4]_q$$

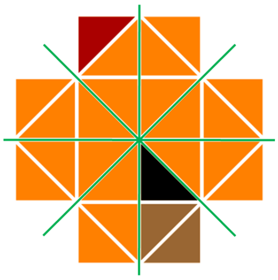
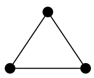


Figure III.3: Bruhat ideal of w with $W_{D(w)}$, $\mathcal{F}_w \cap [e, w]$ and $\mathcal{H}_w$ indicated.

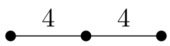
2B Calculations using Computers

A lot of the work on this project has been in building a Wolfram Mathematica notebook [20] which has the ability to do many common computations in Coxeter groups and apply these to the specific questions raised here concerning Bruhat ideals; it has has large classes group elements enumerated and stored in memory to speed up the computation of examples; and it has many functions designed to compute, and in low dimensional cases, visualise examples. We shall use this section to exhibit some of these examples and suggest further questions which can be explored.

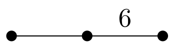
We shall focus on the three irreducible tilings of the plane because these are the easiest to visualise with the aid of a computer. Their Coxeter diagrams are shown in Figure III.4. The diagram automorphism groups are $I_2(3)$, A_1, and $\{e\}$ respectively.



(a) The tiling by equilateral triangles $\tilde{A}_2$.



(b) The tiling of the plane by squares $\tilde{B}_2$.



(c) The tiling of the plane by hexagons $\tilde{G}_2$.

Figure III.4

More examples of Bruhat ideals

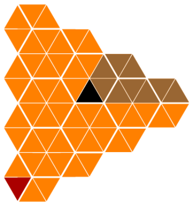
We have computed large numbers of Bruhat ideals in [20, Section 11.5.3], a representative sample is shown in Figure III.5. In each case the fundamental chamber is black and the chamber corresponding to w is red. Those chambers in $\mathcal{F}_w$ are brown. The only non-smooth example we have included here is Figure III.5l. In many of these examples the factorisation of the Poincaré can be seen by decomposing the diagram geometrically. For example in Figure III.5g, the first factor $[2]_q$ comes from the the decomposition of $[e, w]$ into the brown and orange regions. The other two factors come from decomposing the brown region as 4 copies (hence $[4]_q$) of the 2-chain consisting of the fundamental chamber and the chamber directly below it (hence $[2]_q$).

There are some cases however where the geometric decomposition shown is *incompatible* with the factorisation of $P_w(q)$ into q-numbers. Consider Figure III.5b, the length generating function of the chambers in the brown region is $1 + q + 2q^2 + q^3 + q^4$, which is palindromic, but does not factorise into q-numbers. To interpret the factorisation given, think of the first factor $[3]_q$ as coming from acting on

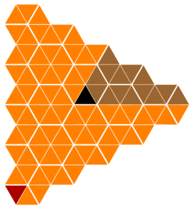
$[e, w]$ by the cyclic group of order three with fundamental domain the union of the brown region and its reflection in its short side. This larger region now decomposes into four copies of three chambers which gives the factors $[3]_q[4]_q$.

(a) $\tilde{A}_2$, $w = 13123$, $P_w(q) = [2]_q[3]_q[3]_q$

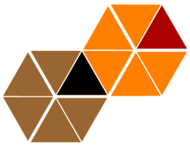
(b) $\tilde{A}_2$, $w = 1312312$, $P_w(q) = [3]_q[3]_q[4]_q$



(c) $\tilde{A}_2$, $w = 131231231$, $P_w(q) = [3]_q[4]_q[5]_q$



(d) $\tilde{A}_2$, $w = 13123123123$, $P_w(q) = [3]_q[5]_q[6]_q$



(e) $\tilde{A}_2$, $w = 2131$, $P_w(q) = [2]_q[2]_q[3]_q$

(f) $\tilde{G}_2$, $w = 212312121$, $P_w(q) = [2]_q[4]_q[6]_q$

(g) $\tilde{G}_2$, $w = 21213$, $P_w(q) = [2]_q[2]_q[4]_q$

(h) $\tilde{G}_2$, $w = 131213$, $P_w(q) = [2]_q[3]_q[4]_q$

(i) $\tilde{B}_2$, $w = 121313$, $P_w(q) = [2]_q[3]_q[4]_q$

(j) $\tilde{B}_2$, $w = 1212312$, $P_w(q) = [2]_q[4]_q[4]_q$

(k) $\tilde{B}_2$, $w = 2123121$, $P_w(q) = [2]_q[4]_q[4]_q$

(l) $\tilde{B}_2$, $w = 121231231$, $P_w(q) = [2]_q[2]_q[4]_q \times (1 + q^2 + q^3 + q^4)$

Figure III.5: Some Bruhat ideals. In each case, the group, element, and Poincaré polynomial are given. For simplicity we replace the generator s_i with i when giving the group elements.

The set of smooth elements

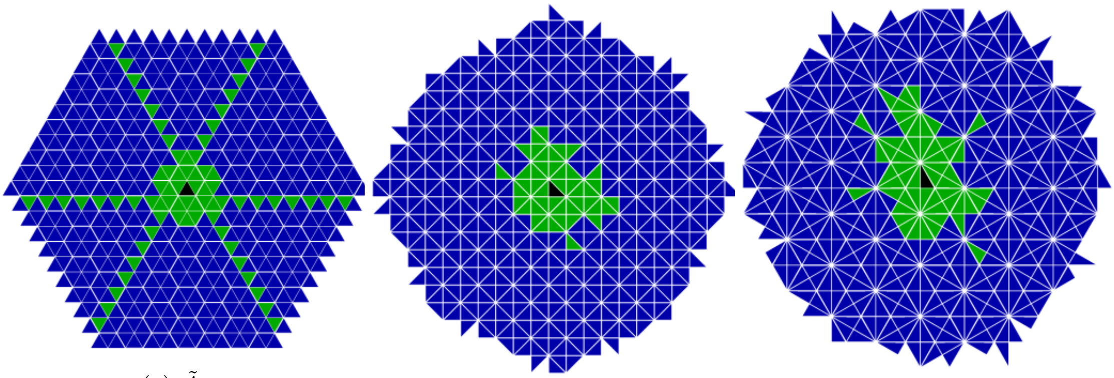
We have computed all smooth elements in $\tilde{A}_2$, $\tilde{B}_2$, and $\tilde{G}_2$ which have length at most 20 [20, Section 11.5.2], these elements are shown in Figure III.6. Note first that as we expect from Theorem III.2 the diagram automorphisms for each group correspond to symmetries in these pictures. These pictures are also invariant under the map $w \mapsto w^{-1}$, but this does not have a simple geometric interpretation (i.e. not as an isometry of the underlying space), so it cannot be seen. We also note that in each case, the elements which have small length are all smooth. We can make this precise.

Proposition III.3. *[5, Lemma 2.7.3 and Corollary 2.7.8] Let (W, S) be a Coxeter system, and $w \in W$ with $l(w) \leq 3$, then w is smooth.*

This is because there are only very few possibilities for what the Bruhat graph can look like if $[e, w]$ is small. It is also clear from our examples that this cannot be improved upon in general. The final observation is that with the exception of $\tilde{A}_2$ there appear to be only finitely many smooth elements, and even in the case of $\tilde{A}_2$ there is just one easily characterisable infinite family of smooth elements up to diagram automorphisms. A few of the Bruhat ideals for these are shown in Figures III.5a–III.5d.

One heuristic way to understand why this may be the case (which necessarily fails in the case of $\tilde{A}_2$) is as follows: we can imagine $[e, w]$ as approximating a ball of radius $l(w)$ centred on the fundamental chamber. Intersect this with the region denoted $\mathcal{F}_w$, the fundamental chamber of the finite automorphism group of $[e, w]$ generated by $D(w)$. It is well known that the fundamental chamber of a finite Coxeter group is a simplicial half-cone (possibly crossed with $\mathbb{R}^k$ for some k), see for example [8, Theorem I.5C]. In particular we would expect $\mathcal{F}_w \cap [e, w]$ to contain more longer elements than shorter ones, so would not be palindromic. This picture appears to become more accurate as $l(w)$ increases, see [20, Section 11.5.3]. Some more precise study of these ideas can be found in [7].

Question 4. *With the exception of cases like $\tilde{A}_2$, is it true that (W, S) only has finitely many smooth elements for any Coxeter system? Can the existence of these simple infinite families of elements be characterised?*



(a) $\tilde{A}_2$ (b) $\tilde{B}_2$ (c) $\tilde{G}_2$

Figure III.6: The chambers corresponding to smooth elements are coloured green, and non-smooth elements blue. The fundamental chamber is black.

The sets of exponents

The final thing we shall discuss is the set of exponents associated to a smooth element (which we know is unique by Lemma III.2). What is noticeable in Figure III.5, but even more striking when a large number of examples is computed (see [20, Section 12.2.1]), is that the number of exponents appears to be bounded by the rank of the group. This is not wholly surprising given the partial factorisation of the Poincaré polynomial given by Lemma III.4. Indeed we know that this is the case in finite types discussed in Chapter II because of the way the exponents were defined in terms of the inversion graph.

Conjecture 2. *Let (W, S) be a Coxeter system which satisfies the q-factorisation property, and let $w \in W$ be smooth. Fix a reduced word for w, and let $S(w)$ be the set of generators in S which appear in this reduced word (it is well known that this set does not depend on the choice of reduced word). Then w in fact lives in the subgroup $W_{S(w)}$, and we conjecture the number of exponents of w is $\#S(w) = \mathrm{Rank}(W_{S(w)})$.*

III.3 Final Remarks

The work of V. Lakshmibai and B. Sandhya [15] later generalised by S. Billey et al. [1, 3, 4] shows that the rational smoothness of Schubert varieties can be characterised by combinatorial properties of the associated Weyl group elements. S. Oh et al. were able to reinterpret these results in the geometry of hyperplane arrangements associated to the Weyl group [17, 18]. None of these results appear to use the fact that the root system associated to the Weyl group is crystallographic in any essential way, so it is natural to ask whether these results generalise to all Coxeter groups. While the question was posed in [18], this appears to be the first attempt to study geometric criteria for the Bruhat ideal to be palindromic in this general setting. The discussion here is very speculative, nevertheless there are many interesting questions to be asked; moreover it is clear that there is a natural pairing between the geometry of the Coxeter group and the Bruhat ideal, and in particular the factorisation of the Poincaré polynomial. Although this final chapter contains little in the way of concrete proofs, we hope to have laid out the first steps of a natural and fruitful path towards answering these questions. We also hope that these ideas can be developed to produce a uniform proof of the known results, which so far have only been proved on a case by case basis.

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