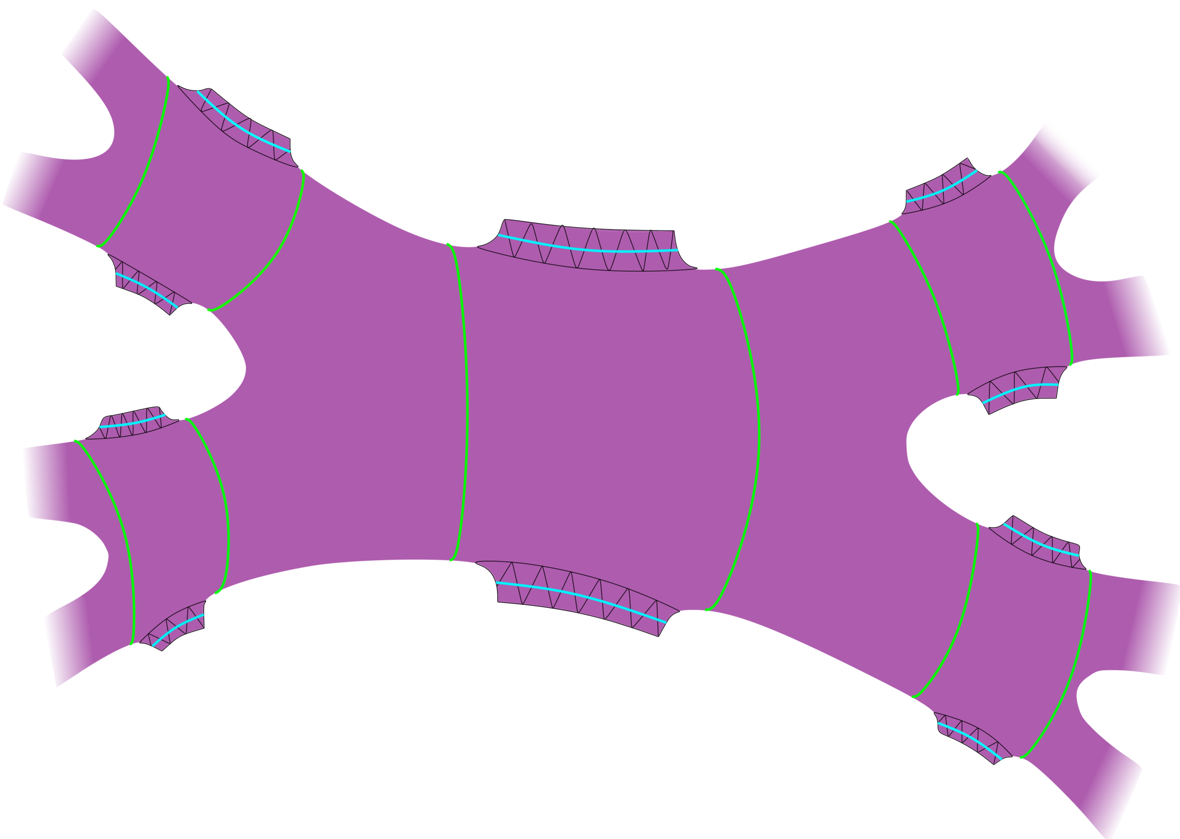
London School of Geometry and Number Theory

Introduction to the Accessibility of Groups

David Sheard



Supervised by
Dr. Larsen Louder

29th June 2018

Abstract

The accessibility of groups is concerned with how groups can be decomposed as free products with amalgamation and HNN extensions. This report aims to be a short introduction to this subject, with the main prerequisites being just a first course in groups and in topology. The main tool is Bass-Serre theory which relates to the way groups act on trees. We motivate and outline this in the first chapter, which for the sake of brevity is necessarily short on detailed proofs. The second chapter discusses generalities of the structure of groups acting on trees and accessibility. We use the second half of the report to discuss some specific results and tools in the study of accessibility in detail. Chapter III is devoted entirely to proving Dunwoody's Theorem on the accessibility of finitely presented groups. Chapter IV discusses the specific case of Coxeter groups which have particularly nice properties with regard to accessibility. In these last two chapters we have endeavoured to prove, or at least justify all of the results. The author would like to thank Dr. Larsen Louder for his help and guidance throughout supervising this project, as well as Joe MacColl for his help understanding M.J. Dunwoody's proof of J.R. Stallings' Theorem given in Chapter III.

Contents

I Introduction

The field of geometric group theory is concerned with studying finitely generated groups by looking at their actions on topological and geometric spaces. The focus here is studying the way in which they act on simplicial trees, from which we can study how a group can be decomposed as a free product of subgroups, as well as more complicated generalisations of such decompositions. We shall begin the introduction by motivating this approach to geometric group theory with some well-known results from topology. We shall give a brief introduction of Bass-Serre theory which is the main tool in what follows. The approach is to use the action of a group on a tree in order to decompose it in terms of the stabilisers of the vertices and edges of the tree.

I.1 Motivation

We shall assume the reader is familiar with expressing groups via presentations with generators and relations, say $G = \langle A \mid R \rangle$. Recall that the free product of two groups is constructed by taking all generators and relations which appear in each group, so if $H = \langle A' \mid R' \rangle$ then the free product of G and H is $G * H = \langle A \cup A' \mid R \cup R' \rangle$. We define two more group constructions of a similar flavour.

Definition I.1. Let $G_1 = \langle A_1 \mid R_1 \rangle$, $G_2 = \langle A_2 \mid R_2 \rangle$, and H be groups with homomorphisms $f_1 : H \mapsto G_1$ and $f_2 : H \mapsto G_2$, then the **amalgamated product** of G_1 and G_2 over H (with respect to f_1 and f_2) denoted by $G_1 *_H G_2$ is the group with presentation

$$\langle A_1 \cup A_2 \mid R_1 \cup R_2,\ f_1(h) = f_2(h)\ \forall h \in H \rangle.$$

We shall usually be in the situation that f_1 and f_2 are injective so we can view H as a common subgroup of G_1 and G_2, and so $G_1 *_H G_2$ corresponds to the free product of G_1 and G_2 to which we have added relations to identify the subgroups isomorphic to H. Indeed taking $H = \{1\}$ clearly reduces to the case of a free product. The second construction is due to G. **H**igman, B. **N**eumann, and H. **N**eumann.

Definition I.2. Let $G = \langle A \mid R \rangle$ be a group with subgroup H, and take $f : H \mapsto G$ an injective homomorphism (in particular $f(H)$ is a subgroup of G isomorphic to H). The **HNN extension** of G by H (with respect to f) denoted[1] by $G*_{H,t}$ is the group with presentation

$$\langle A, t \mid R,\ tht^{-1} = f(h)\ \forall h \in H \rangle$$

where t is a formal generator we introduce called the **stable letter**.

In this case we introduce a new generator and relations to force H and $f(H)$ to be conjugate in $G*_{H,t}$. Note that the most trivial amalgamated product $\{1\} *_{\{1\}} \{1\}$ just gives the trivial group, however $\{1\}*_{\{1\},t} = \langle t \mid \rangle \cong \mathbb{Z}$ is not trivial. In either of these constructions we say that the resulting group **splits** over the subgroup H with **factors** G_1 and G_2, or just G respectively.

These constructions arise naturally when studying the fundamental group of topological spaces. Let X be a topological space, recall that its fundamental group is the group of homotopy classes of based loops with group operation the concatenation of loops. We might as well assume X is path connected so that we do not need to worry about base points, then we denote this group $\pi_1(X)$. The following two theorems are well-known.

Theorem I.3 (van Kampen)**.** *Let Y, X_1, and X_2 be non-empty path connected topological spaces, and let $f_1 : Y \mapsto X_1$ and $f_2 : Y \mapsto X_2$ be homeomorphisms such that $f_i(Y) \subset X_i$ is open. Let X' be the space constructed from X_1, X_2 and $Y \times [0,1]$ by glueing the ends of $Y \times [0,1]$ to X_1 and X_2 via the maps f_1 and f_2 respectively. Then*

$$\pi_1(X') = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$

where the group homomorphisms are the obvious ones induced by the maps f_i.

[1]There are number of conflicting notations for an HNN extension. As with amalgamated products, we suppress the homomorphism f in the notation.

Sketch proof. For simplicity assume the induced homomorphisms are injective (justifying the theorem without this condition takes a bit more work). Take loops γ_i in X_i for $i = 1, 2$, each of which represents a non-trivial element in $\pi_1(X_i)$. By injectivity we expect these loops to represent non-trivial elements in $\pi_1(X')$. We need to account for the possibility that γ_1 is homotopy equivalent to γ_2 in X', in which case it is clear that there is a loop $\gamma \subset Y \times [0,1]$ homotopy equivalent to both γ_1 and γ_2. Hence in $\pi_1(X')$ we need to ensure that the classes of γ_1, γ_1, and γ coincide, but this is precisely what the amalgamated product does for us. ■

Theorem I.4. *Let X be a (path connected) topological space, and $Y \subset X$ an open path connected subspace. Let $f : Y \mapsto X$ be a homeomorphism and define a new space X' by glueing a copy of $Y \times [0,1]$ to X such that $Y \times \{0\}$ is identified with Y and $Y \times \{1\}$ is attached according to the map f. Then*

$$\pi_1(X') = \pi_1(X) *_{\pi_1(Y),t}$$

where the homomorphism is induced by f in the obvious way.

The justification for this follows similar reasoning to van Kampen's theorem. The stable letter t can be thought of as a loop which goes around the "handle" we have added thus introducing genus. We can now use these results to calculate the fundamental groups of some spaces very easily.

Example I.5. It is clear from the definition of the fundamental group that $\pi_1(X)$ depends only on the homotopy equivalence class of X, and that $\pi_1(\text{point}) = \{1\}$. $[0,1] \simeq \{\text{point}\}$, hence $\pi_1([0,1]) = \{1\}$. We can get S^1 by identifying the (open neighbourhoods of the) endpoints of $[0,1]$ and then the second theorem says $\pi_1(S^1) = \{1\} *_{\{1\},t} \cong \mathbb{Z}$. The cylinder $[0,1] \times S^1 \simeq S^1$ is homotopy equivalent to a circle, so it has fundamental group $\mathbb{Z}$ as well.

We can construct S^2 by gluing two discs D along a neighbourhood of their boundary (a cylinder) to see that $\pi_1(S^2) = \{1\} *_{\mathbb{Z}} \{1\} = \{1\}$. Note we get the trivial group because the homomorphisms induced by inclusion are not injective. We can also calculate the fundamental group of the torus T^2 by identifying the ends of a cylinder to get $\pi_1(T^2) = \mathbb{Z} *_{\mathbb{Z},t} \cong \mathbb{Z}^2$.

As a final example we shall compute the fundamental group of a more complicated space: consider the wedge of two circles crossed with an interval. This gives two cylinders which meet tangentially in a line. Identify the two ends of this double cylinder after making k half-twists to get a space X_k, what is $\pi_1(X_k)$? If k is even then X_k can be thought of a torus wrapping around another torus, intersecting in a single line which is a torus knot (q.v. Figure I.1a), so the fundamental group is an amalgamated product

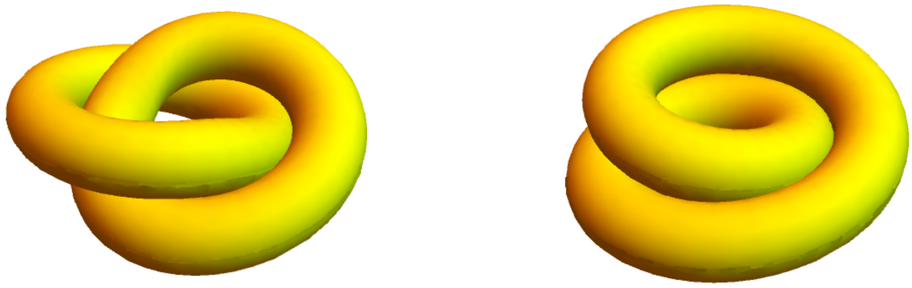
$$\pi_1(X_k) = \langle a, b \mid ab = ba \rangle *_{\langle x \rangle} \langle c, d \mid cd = dc \rangle$$

with homomorphisms $f_1 : x \mapsto b$ and $f_2 : x \mapsto c^k d$. After some manipulation we see $\pi_1(X_k) = \mathbb{Z} \times F_2$.

If k is odd then we end up with a space homotopic to a Möbius band whose single edge is replaced by a torus (q.v. Figure I.1b). In this case $\pi_1(X_k)$ can be obtained as an HNN extension of the fundamental group of the wedge of two circles. $\pi_1(S^1 \vee S^1) = \pi_1(S^1) *_{\pi_1(\text{point})} \pi_1(S^1) = F_2$ the free group on two generators $\{x, y\}$, so

$$\pi_1(X_k) = F_2 *_{F_2,t}$$

with homomorphism $f : x \mapsto y$ and $y \mapsto x$. Whence $\pi_1(X_k) = \langle x, y, t \mid txt^{-1} = y \rangle$.

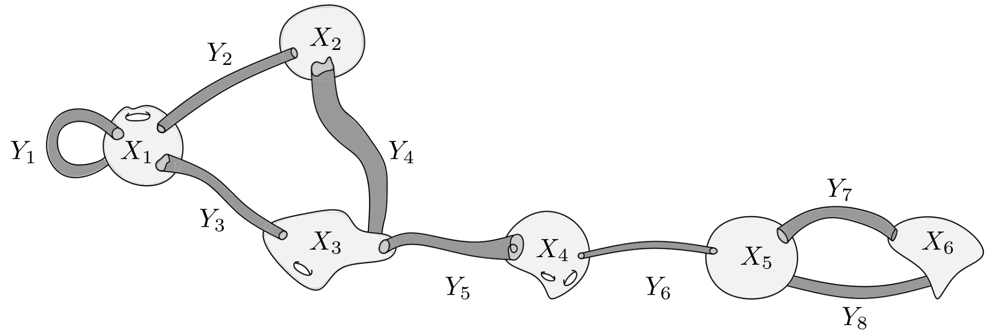


(a) (b)

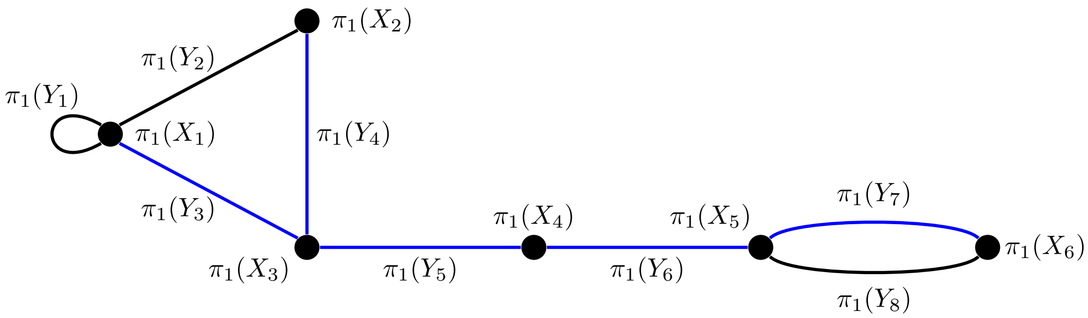
Figure I.1

The starting point for the study of groups by looking at spaces in this way is the observation that given any group G there is a space X such that $G \cong \pi_1(X)$. In the case that G is finitely presented, X can be taken to be a 2–dimensional CW-complex as follows. Take a bouquet (wedge product) of oriented circles, one for each of the k generators. This has fundamental group the free group on k generators. Each relation in G is a word in the generators and their inverses, attach a disc to the bouquet of circles according to each of these relations thus trivialising the corresponding loop in the fundamental group. This gives the required space X which is called the presentation complex for G, whose universal cover is the Cayley 2-complex of G.

We want to study the way a group G decomposes as an arbitrary number of amalgamated products and HNN extensions simultaneously. The way to think of this topologically is as a *graph of spaces*, where the "vertices" are the spaces X_1, X_2, and X in the two theorems, and the "edges" are the handles $Y \times [0, 1]$. An example of such a graph of spaces is shown in Figure I.2a. We have also exhibited the underlying graph with edges and vertices labelled by the fundamental group of the corresponding space. Note that for each edge group we have homomorphisms into the groups labelling the endpoints of the edge.



(a) A graph of spaces in which the spaces are labelled X_i, and the edges $Y_j \times [0, 1]$ are labelled Y_i.



(b) The underlying graph labelled with the fundamental groups. A maximal subtree is coloured blue.

Figure I.2

We can choose a maximal tree in the underlying graph (say the one shown in blue), and collapse the corresponding edges in the graph of spaces by appropriate homotopies so as to view

$$\bigcup_i X_i \cup \left(\left(\bigcup \{Y_3, Y_4, Y_5, Y_6, Y_7\}\right) \times [0, 1]\right)$$

as a single space X with three "handles". On the level of fundamental groups this collapsing of edges corresponds to series of amalgamated products which we write out explicitly for the sake of completeness[2].

$$\pi_1(X) = (\pi_1(X_1) *_{\pi_1(Y_3)} \pi_1(X_3) *_{\pi_1(Y_4)} \pi_1(X_2)) *_{\pi_1(Y_5)} \pi_1(X_4) *_{\pi_1(Y_6)} \pi_1(X_5) *_{\pi_1(Y_7)} \pi_1(X_6)$$

This gives us a new graph of spaces which is a bouquet of circles as shown in Figure I.3. Collapsing the edges of this by appropriate homotopies corresponds to a series of HNN extensions on the level of

[2]Considering the definition of amalgamated products we see that the order in which the edges are collapsed does not matter, the same holds for HNN extensions.

fundamental groups, so we end up with a single space X' whose fundamental group is

$$\pi_1(X') = ((\pi_1(X) *_{\pi_1(Y_1),t_1}) *_{\pi_1(Y_2),t_2}) *_{\pi_1(Y_8),t_8}$$

Hence we have a correspondence between a complicated decomposition of a space as a graph of spaces, and an equally complicated decomposition of its fundamental group as a *graph of groups*, or equivalently as a series of amalgamated products and HNN extensions.

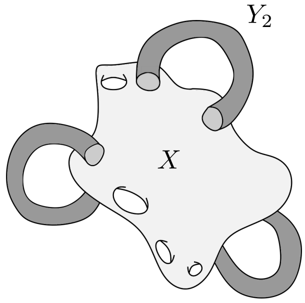


Figure I.3: The graph of spaces after collapsing a maximal subtree.

Bass-Serre theory formalises all of this, as well as gets round the problem of possibly having complicated decompositions of "simple" spaces which corresponds to a trivial decomposition of the fundamental group (as was the case in Example I.5 when decomposing S^2 gave $\{1\} = \{1\} *_{\mathbb{Z}} \{1\}$). What Bass and Serre were able to show is that we can in fact forget about graphs of spaces (although they are a useful intuitive tool), and just consider the action of a group G on a tree T. Then $G\backslash T$, labelled by the stabilisers of the vertices and edges, is a graph of groups corresponding to a decomposition of G as a series of amalgamated products and HNN extensions analogous to the above.

I.2 Bass-Serre Theory

Details of the material in this section can be found in [20].

2A Graphs and Group Actions

We begin with the formulation of a combinatorial graph due to Serre.

Definition I.6. A **graph** Γ consists of two sets $V = \mathrm{Vert}(\Gamma)$ and $E = \mathrm{Edge}(\Gamma)$ (whose elements are called the **vertices** and **edges** of Γ respectively), together with maps

$$E \mapsto V \times V : e \mapsto (o(e), t(e)) \qquad\qquad E \mapsto E : e \mapsto \overline{e}$$

which satisfy $e \neq \overline{e}$, $\overline{\overline{e}} = e$, and $o(e) = t(\overline{e})$. For an edge e, the vertices $o(e)$ and $t(e)$ are called the **origin** and **terminus** of e respectively, and collectively as the **extremities** or **endpoints** of e.

Such a graph can be realised topologically in the usual way, with the vertices given as points, and edge pairs $\{e, \overline{e}\}$ as arcs. An *orientation* of Γ amounts to a choice of $E_+ \subset E$ such that $E = E_+ \sqcup \overline{E_+}$. We shall assume the reader is familiar with the basic notions and terminology of graph theory which carry over to this formulation. In particular a **tree** is a connected graph which contains no cycles (equivalently a graph whose topological realisation is connected and simply connected).

Definition I.7. A group G **acts** on a graph Γ (on the left) if it acts on the sets V and E such that the structure of Γ is preserved. Then we call Γ a G-**graph**. We say that G acts **freely**, if the action is free on the set of vertices (i.e. vertex stabilisers are trivial).

An equivalent way of thinking of this is that a G-graph is a graph Γ together with a representation $G \mapsto \mathrm{Aut}(\Gamma)$. Clearly we can make any graph Γ into a G-graph by making G act trivially. A more interesting class of examples comes from Cayley graphs.

Definition I.8. Let G be a group and S a subset of G, then the **Cayley graph** of G relative to S, denoted $\Gamma(G,S)$, is the oriented graph which has vertex set $V = G$, and $E_+ = G \times S$ with $o(g,s) = g$ and $t(g,s) = gs$ for each edge $(g,s) \in G \times S$.

Often it is useful to take S to be a finite set of generators of G (if such a set exists) in which case $\Gamma(G,S)$ is connected and cycles correspond to relations in the group. In any case G acts on $\Gamma(G,S)$ freely. As a first application of studying the way groups act on graphs, and to yet further motivate the approach discussed below, we study the structure of subgroups of free groups by examining the actions of these groups on trees.

Proposition I.9. *[20, Proposition I.15] Let G be a group and S a subset of G, then $\Gamma(G,S)$ is a tree if and only if G is free with free basis S.*

Proof. $\Gamma(G,S)$ is a tree if and only if it is connected and contains no cycles. Connectedness is equivalent to S generating G, and the fact that there are no cycles implies that G has no relations, hence it is free. We just need to establish the equivalence of cycles in the Cayley graph, and relations in the group. Let C be a cycle with length $n > 1$ and no back-tracking, then we may assume 1 is a vertex in C (if not, pick a vertex g in C, and act by g^{-1} to give a cycle $g^{-1}C$ which contains 1). Going around this cycle starting from a vertex adjacent to 1 gives a sequence of elements $s_1, s_1s_2, s_1s_2s_3, \ldots, s_1s_2\cdots s_n$. Then necessarily $s_1s_2\cdots s_n = 1$ in G, so G has some non-trivial relation. Conversely it is clear that if we have a relation, then this defines a cycle in $\Gamma(G,S)$. ■

Theorem I.10. *[20, Theorem I.4] A group G is free if and only if there is a tree on which it acts freely.*

Proof. By the previous proposition, if G is free, then it acts freely on $\Gamma(G,S)$ (for S some free basis), which is a tree. Conversely let T be a tree on which G acts freely. Passing to the topological realisation of T, since it is simply connected T is the universal cover of $G\backslash T$ (the topological quotient is always defined, even if the quotient graph is not — see below). Hence G can be identified with $\pi_1(G\backslash T)$, but since $G\backslash T$ is homotopy equivalent to a bouquet of circles, G must be free. ■

This theorem now follows immediately.

Theorem I.11 (Schreier's Theorem)**.** *[20, Theorem I.5] Every subgroup of a free group is free.*

Proof. Let G be free, then it acts freely on a tree. If H is a subgroup of G, then it necessarily also acts freely on the same tree, hence is itself free. ■

Given a G-graph Γ, we say that G acts **without inversion** if there is no element $g \in G$ together with an edge e such that $g.e = \overline{e}$. It is easy to see that G acts without inversion if and only if there is some orientation E_+ of Γ which is preserved by the action of G. When we have such an action, then the quotient $G\backslash\Gamma$ satisfies the definition of a graph given above. If G acts on Γ *with* inversion, then we can easily obtain a graph from Γ on which G acts without inversion by equivariantly subdividing all edges which are inverted, introducing new orbits of vertices consisting of the midpoints of all problematic edges. Henceforth, unless stated otherwise, we shall assume that if we have a G-graph then G acts without inversion.

2B Graphs of Groups

We want to formalise the notion above of a series of amalgamated products and HNN extensions via the fundamental group of a graph of groups.

Definition I.12. A **graph of groups** $\mathcal{G} = (G,\Gamma)$ consists of a connected graph Γ and groups G_v for each $v \in V$ and G_e for each $e \in E$ such that $G_e = G_{\overline{e}}$, along with *injective* homomorphisms $G_e \hookrightarrow G_{t(e)}$.

This is the same as the example shown in Figure I.2b except with the added injectivity condition. We want to define an analogue of $\pi_1(X')$ in that example. First we shall define the group $F(G,\Gamma)$ associated to a graph of groups $\mathcal{G} = (G,\Gamma)$ as the group generated by the vertex groups of $\mathcal{G}$ together with the elements $e \in E$ subject to relations

$$\overline{e} = e^{-1}$$

$$ef(g)e^{-1} = \overline{f}(g)\ \forall g \in G_e$$

where $f : G_e \hookrightarrow G_{t(e)}$ and $\overline{f} : G_{\overline{e}} \hookrightarrow G_{t(\overline{e})}$.

Definition I.13. Let $\mathcal{G} = (G, \Gamma)$ be a graph of groups, and T' a maximal subtree of Γ. The **fundamental group** of $\mathcal{G}$ is the group $\pi_1(G, \Gamma)$ obtained as the quotient of $F(G, \Gamma)$ by the subgroup normally generated by the generators $e \in \mathrm{Edge}(T')$.

This is directly analogous to what we did in the first section calculating the fundamental group of a graph of spaces. It can be shown that the fundamental group does not depend on the choice of maximal subtree T'. Note that if we quotient out by e then the relation in $F(G, \Gamma)$ reduces to $f(g) = \overline{f}(g)$ which was the relation in an amalgamated product; otherwise $ef(g)e^{-1} = \overline{f}(g)$ is the relation in an HNN extension with e the stable letter. The graph of groups corresponding to a single amalgamated product consist of a segment with two vertices, and an HNN extension corresponds to a single vertex with a loop. Henceforth, by an abuse of notation, we shall not distinguish an edge e from its reverse $\overline{e}$. The following structure theorem for the fundamental group of a graph of groups is key to the structure theorem of groups acting on trees in the next section.

Theorem I.14. *[20, Theorem I.2] Let $\mathcal{G} = (G, \Gamma)$ be a graph of groups, then the canonical homomorphisms $G_v \mapsto \pi_1(G, \Gamma)$ and $G_e \mapsto \pi_1(G, \Gamma)$ defined by sending generators to their images in $\pi_1(G, \Gamma)$ (where G_v is a vertex group and G_e is an edge group), are injective.*

Remark I.15. If all the vertex and edge groups are trivial then $\pi_1(G, \Gamma)$ is isomorphic to the fundamental group of the underlying graph Γ, thus in general by mapping each vertex and edge group to $\{1\}$ we get a surjective homomorphism $\pi_1(G, \Gamma) \mapsto F_{\beta_1}$ onto the free group on β_1 generators, where β_1 is the first Betti number of Γ. On the other hand if $\mathcal{G}$ arises from a graph of spaces which is a decomposition of a space X' then $\pi_1(G, \Gamma) = \pi_1(X')$.

We are most interested in the case when $\mathcal{G}$ arises from the action of a group on a tree as follows. Let T be a G-tree and let $G\backslash T$ be the quotient graph. The vertices and edges of $G\backslash T$ are orbits of vertices and edges of T under the G-action. Choose a representative from each orbit and label the corresponding vertex or edge in $G\backslash T$ by the stabiliser of this representative. Suppose v and v' are two vertices of T which are in the same orbit, and $gv = v'$ for some $g \in G$, then $G_{v'} = gG_v g^{-1}$ where $G_v = \mathrm{Stab}(v)$ and $G_{v'} = \mathrm{Stab}(v')$, so these vertex and edge labels are well-defined only up to conjugacy (similarly for edge groups). In order to define the homomorphisms of the edge groups into the vertex groups we need the following result.

Proposition I.16. *[20, Proposition I.14] Let Γ be a connected G-graph. Every subtree T' of $G\backslash T$ lifts to a subtree of T.*

Proof. Consider the set of all subtrees of Γ which map injectively into $G\backslash\Gamma$ ordered by containment, and let T'_0 be a maximal such subtree. If T'_0 does not map onto T' then there is some edge e of T' not represented by an edge of T'_0. We can lift this to an edge $\tilde{e}$ in Γ which has an endpoint in T'_0, and we claim $T'_1 = T'_0 \cup \tilde{e}$ is a tree which is therefore mapped injectively into T' contradicting the maximality of T'_0. Indeed if T'_1 were not a tree, then both endpoints of $\tilde{e}$ are in T'_0 and thus e must form part of a cycle in T' which is not possible. ■

Returning to G acting on a tree T, we define the homomorphisms in the resultant graph of groups as follows. Let T' be a maximal tree in $G\backslash T$, and let $\tilde{T'}$ be a lift of T' in T which is a **tree of representatives** for $G\backslash T$. T' contains all vertices of $G\backslash T$, so take the vertex groups of $G\backslash T$ to be the stabilisers of the corresponding vertex in $\tilde{T'}$. Let e be an edge of T' whose lift is $\tilde{e}$, since G acts without inversion if $g \in G$ fixes $\tilde{e}$ then it fixes $o(\tilde{e})$ and $t(\tilde{e})$ hence we can take the homomorphisms to be simple inclusion $G_{\tilde{e}} \hookrightarrow G_{o(\tilde{e})}$ and $G_{\tilde{e}} \hookrightarrow G_{t(\tilde{e})}$.

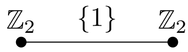
We are now left with the edges of $G\backslash T$ not contained in T'. Let e now be such an edge, both of its endpoints are in T' so there are lifts $\tilde{e}$ and $\tilde{e}'$ of e such that $g\tilde{e} = \tilde{e}'$ for some $g \in G$, $o(\tilde{e}) \in \tilde{T'}$ represents $o(e)$ and $t(\tilde{e}') \in \tilde{T'}$ represents $t(e)$. Then as before we have inclusions $G_{\tilde{e}} \hookrightarrow G_{o(\tilde{e})}$ and $G_{\tilde{e}'} \hookrightarrow G_{t(\tilde{e}')}$ and $G_{\tilde{e}'} = gG_{\tilde{e}}g^{-1}$. Therefore take $G_{\tilde{e}}$ as the edge group of e, with inclusion into one vertex group, and the conjugate by g of inclusion into the other vertex group. These homomorphisms are all injective as required thus we have defined a valid graph of groups.

Remark I.17. It is clear that the requirement that T is a tree is redundant in this definition, we just need it to be connected. The reason we restrict ourselves to trees is made plain be the structure theorem in the next section.

2C Examples and the Structure Theorem

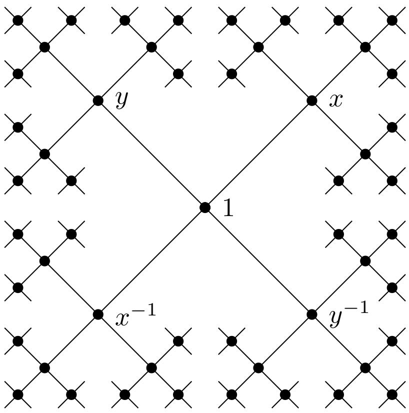
Before we proceed to the structure theorem for groups acting on trees we give some examples of graphs of groups arising from group actions on trees as described above.

Example I.18. Consider the real line as a graph, with vertex set the set of integers, on which the infinite dihedral group D_∞ acts by translations and reflections. The edge stabilisers are trivial, and vertex stabilisers are copies of $\mathbb{Z}_2$, so the graph of groups looks like

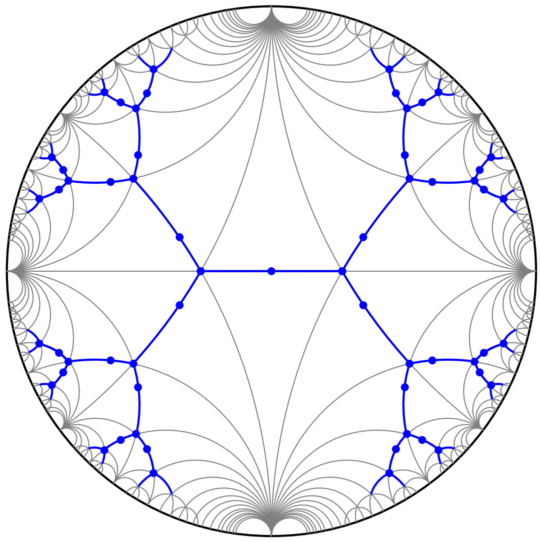


Example I.19. The free group F_2 on two generators x and y acts freely on the Cayley graph $\Gamma(F_2, \{x, y\})$ as described in Definition I.8. The Cayley graph and graph of groups are shown in Figures I.4a and I.4c.

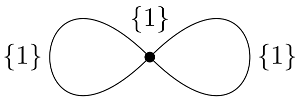
Example I.20. It is well-known how $SL_2(\mathbb{Z})$ acts on the hyperbolic plane. This gives a tiling of the hyperbolic plane by the fundamental domain of the action as shown by the grey lines in Figure I.4b. This induces an action on an infinite tree (shown in blue), with fundamental domain an interval. The stabilisers can be computed and the corresponding graph of groups is shown in Figure I.4d.



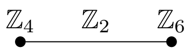
(a) The Cayley graph of F_2.



(b) The action of $SL_2(\mathbb{Z})$ on the Poincaré disc with induced action on an embedded tree.



(c) The graph of groups corresponding to the action of F_2 on its Cayley graph.



(d) The graph of groups corresponding to the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}^2$.

Figure I.4: Examples of graphs of groups arising from group actions on trees.

The main tenent of Bass-Serre theory is the following structure theorem for groups acting on trees. The case when the associated graph of groups has a single edge was proved by Serre, the general case by Bass.

Theorem I.21. *[20, Theorem I.13] Let G be a group acting on a graph Γ with associated graph of groups $\mathcal{G} = (G, \Gamma')$ and let $\phi : \pi_1(G, \Gamma') \mapsto G$ be the homomorphism induced by the inclusion of the vertex stabilisers of $\mathcal{G}$ into G. Then ϕ is an isomorphism if and only if Γ is a tree.*

One can easily check the veracity of the theorem in Examples I.18 and I.19 above, the theorem implies that $SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ from Example I.20. An easy example of the contrapositive of the statement is to consider the trivial group acting trivially on any graph Γ containing a cycle. Then the graph of groups has Γ as its underlying graph, all stabilisers are trivial, and the fundamental group of this graph of groups is some free group with positive rank, in particular it is not the trivial group.

The proof of the full theorem is too long to include here, but we can sketch the proof if we limit to the case that $G\backslash\Gamma = \overset{G_v}{\bullet}\overset{G_e}{\rule[0.5ex]{3em}{0.4pt}}\overset{G_u}{\bullet}$ is just a segment, so we are representing G as a single amalgamated product of the vertex groups over the edge group. The theorem follows from two lemmata.

Lemma I.22. *[20, Lemma I.2]* Γ *is connected if and only if* G *is generated by* $G_v \cup G_u$ *(so in particular* ϕ *is surjective).*

Proof. Let $\tilde{T} = \overset{v}{\bullet}\overset{e}{\rule[0.5ex]{2em}{0.4pt}}\overset{u}{\bullet}$ be a lift of $G\backslash\Gamma$ in Γ, and let Γ' be the connected component of Γ containing $\tilde{T}$. Denote by $G_{\Gamma'}$ the elements $g \in G$ such that $g\Gamma' = \Gamma'$, and let G' be the subgroup of G generated by $G_v \cup G_u$. For $h \in G_v \cup G_u$, $h\tilde{T}$ and $\tilde{T}$ share a common vertex so $h\tilde{T} \subset \Gamma'$, hence $h\Gamma' = \Gamma'$ and $h \in G_{\Gamma'}$. On the other hand $G'\tilde{T}$ and $(G - G')\tilde{T}$ are disjoint subgraphs of Γ whose union is Γ, so $G'\tilde{T}$ contains Γ'. Hence $G_{\Gamma'} = G'$. The graph Γ is connected if and only if $\Gamma = \Gamma'$, i.e. if $G = G_{\Gamma'} = G'$. ∎

Lemma I.23. *[20, Lemma I.3]* Γ *contains no cycles if and only if* ϕ *is injective.*

Sketch Proof. Suppose there is a cycle $(e_0, e_1, \dots, e_k)$ in Γ, which necessarily projects onto the interval $G\backslash\Gamma$ sending alternate vertices to the opposite endpoints of the segment. If e_0 is our chosen lift of the edge $e \in G\backslash\Gamma$ then $e_i = h_i e_0$ for some $h_i \in G$. It is easy to see that $h_i = h_{i-1}g_i$ for $g_i \in G_{o(e_i)}$, and because e_i and e_{i-1} intersect only in a vertex, $g_i \notin G_{e_i}$.

Since $o(e_0) = t(e_k)$ we have $h_0 o(e_0) = h_n o(e_0) = h_0 g_1 \cdots g_k o(e_0)$ so $g_1 \cdots g_k \in G_{o(e_0)}$. Hence we have the sequence of elements $g_i \in G_{o(e_i)} - G_{e_i}$ such that $g_0 g_1 \cdots g_k = 1$ in G, however Theorem I.14 says that $g_0 g_1 \cdots g_k \neq 1$ in $G_u *_{G_e} G_v$, so ϕ is not injective. ∎

If $\mathcal{G}$ is a graph of groups whose fundamental group is isomorphic to G, we say that $\mathcal{G}$ is a **graph of groups decomposition** of G. To finish this chapter we claim that in fact every graph of groups decomposition $\mathcal{G}$ of a group G can be realised via the fundamental group acting on an appropriate tree. We shall do this by constructing a tree $\tilde{\Gamma}$ which is in some sense the universal cover of $\mathcal{G}$ with an action of $\pi_1(\mathcal{G}) = G$ on $\tilde{\Gamma}$ such that $\mathcal{G} = (G, \Gamma) = (G, G\backslash\tilde{\Gamma})$.

$\tilde{\Gamma}$ is called the **Bass-Serre tree** of $\mathcal{G}$ and is defined as follows. The vertices and edges of $\tilde{\Gamma}$ are the disjoint unions over the vertices and edges of Γ of the cosets of the vertex and edge groups.

$$\mathrm{Vert}(\tilde{\Gamma}) := \coprod_{v \in \mathrm{Vert}(\Gamma)} G/G_v \qquad\qquad \mathrm{Edge}(\tilde{\Gamma}) := \coprod_{e \in \mathrm{Edge}(\Gamma)} G/G_e$$

G acts on these sets on the left and it is clear that the stabiliser of the vertex G_v (a lift of $v \in \mathrm{Vert}(\Gamma)$), is the subgroup G_v. To make $\tilde{\Gamma}$ a graph we still need to specify the map $\mathrm{Edge}(\tilde{\Gamma}) \mapsto \mathrm{Vert}(\tilde{\Gamma}) \times \mathrm{Vert}(\tilde{\Gamma})$ (recall we are no longer distinguishing edges e and $\overline{e}$). For any $g \in G$

$$o(gG_e) = gG_{o(e)} \qquad\qquad t(gG_e) = geG_{t(e)}$$

Recall $G = \pi_1(\mathcal{G})$ takes the edges $e \in \mathrm{Edge}(\Gamma)$ as generators. It is an easy exercise to check that these expressions give rise to a well-defined graph, and that G acts without inversion. It is harder to show that $\tilde{\Gamma}$ is a tree [20, Theorem I.12], and has the properties claimed above [20, Theorem I.13]. Moreover it is true that if G acts on a tree T giving a graph of groups $\mathcal{G}$ and $\tilde{\Gamma}$ is the Bass-Serre tree for $\mathcal{G}$, then T and $\tilde{\Gamma}$ are isomorphic as G-trees. This means not only that every graph of groups decomposition of G can be realised by G acting on a tree, but also that there is a unique such realisation up to isomorphism (of G-trees).

Because we have an isomorphism between a group acting on a tree and the fundamental group of the graph of groups arising from that action, we shall spend the rest of the report studying the way groups split by studying their graphs of groups. We shall from now on use the following conventions to simplify our $\mathcal{G}$ of G as representing a G-equivalence class of vertices in the G-tree $\tilde{\Gamma}$. Suppose v is the representative of this equivalence class such that the vertex in $\mathcal{G}$ is labelled by the stabiliser of v, then we shall use V to denote simultaneously the equivalence class of vertices in $\tilde{\Gamma}$, the vertex of $\mathcal{G}$, and the stabiliser group labelling this vertex. Similarly for an edge e of $\tilde{\Gamma}$ we shall label the class, edge, and group in $\mathcal{G}$ by E.

II Accessibility

A graph of groups decomposition of a group G is **trivial** if one of the vertex groups is G itself. Before we can introduce the notion of accessibility we need to discuss how to tell whether a group has any non-trivial splittings at all.

II.1 Conditions for the Existence of Splittings

This first observation is trivial.

Lemma II.1. *[16, Section 2] Suppose G does not admit any surjection $G \mapsto \mathbb{Z}$ (equivalently does not have any quotient isomorphic to $\mathbb{Z}$), then G does not split as an HNN extension over any subgroup.*

Indeed if it did, then the homomorphism which sends the stable letter to 1, and trivialises all other generators contradicts the hypothesis. If G satisfies this lemma, then it can only split as amalgamated product, so if $\mathcal{G}$ is a graph of groups decomposition of G, then the underlying graph must be a tree.

Example II.2. A **Coxeter group** W is a group generated by reflections which acts discretely. Coxeter groups admit a presentation of the form

$$W = \langle S = \{s_1, \ldots, s_k\} \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \ \forall\, 1 \leq i < j \leq k \rangle$$

for $m_{ij} \in \{2, 3, \ldots, \infty\}$, where $m_{ij} = \infty$ means s_i and s_j are unrelated. Since all the generators are torsion, any homomorphism $W \mapsto \mathbb{Z}$ must send S to 0, and hence be trivial.

More generally, the following is an easy observation.

Lemma II.3. *A group G does not admit any surjection onto $\mathbb{Z}$ if and only if it is generated by some subset A consisting of torsion elements.*

1A The Property (FA)

J. Serre introduced the following property for a group G: (FA)[1]

> Whenever G acts on a tree, there is a global fixed point.

It is an easy exercise to show that whenever a finite group acts on a tree it stabilises a vertex or an edge, so all finite groups have the property (FA). In general one can see that if G has the property (FA), and $\mathcal{G}$ is a graph of groups decomposition from the action of G on a tree T, then $\mathcal{G}$ will have a vertex labelled G, hence we have the following.

Theorem II.4. *[20, Theorem I.15] G has the property* (FA) *if and only if the following are satisfied:*

1. *G has no non-trivial splitting as an amalgamated product or HNN extension, and*
2. *G is not the union of an increasing sequence of subgroups (if G is countable, we can replace this with the condition that G is finitely generated).*

We list some properties and consequences which follow immediately.

1. If G is contained in an amalgam $G_1 *_H G_2$ and has the property (FA) then it is contained in a conjugate of G_1 or G_2.
2. If G has the property (FA) then every quotient of G has the property (FA).
3. Let $H \triangleleft G$, if H and G/H have the property (FA), then so does G.

As an example, J. Serre was able to show that $SL_3(\mathbb{Z})$ has the property (FA) and so does not split; this was generalised to all subgroups of finite index by G. Margulis and J. Tits [14, Proposition 2].

[1] From the French *fixe arbre* meaning fixes a tree.

1B Elliptic and Hyperbolic Group Actions

If there is no global fixed point of a group acting on a tree then we can classify the way group elements act on the tree as follows.

Lemma II.5. *[24, Proposition 3.2], [20, Proposition I.25] Let G act on a tree T, and let $g \in G$. Either g fixes a point (in which case we say g acts **elliptically**), or else it fixes no point and there is a unique doubly infinite path (the **axis** of g) which is stabilised by g (in which case we say g acts **hyperbolically**).*

Proof. We consider the action of G on the geometric realisation of T. For two points $x, y \in T$, let $[x, y]$ denote the unique geodesic joining them. Consider the action of an element g and assume it fixes no point, we shall show it fixes an axis.

For v a vertex of T, let m be the midpoint of $[v, gv]$. If $[v, gv] \cup [gv, g^2v] = [v, g^2v]$ then it is clear that $\langle g \rangle$-translates of $[v, gv]$ form an invariant axis and we are done, so assume this is not the case. Then let o be the unique valence three vertex in the subgraph $[v, gv] \cup [gv, g^2v]$ (q.v. Figure II.1). If $d(v, m) \geq d(v, o)$ then g fixes m contradicting our assumption, hence $d(v, m) < d(v, o)$. We claim that $d(m, g^2m) = 2d(m, gm)$ so that $\langle g \rangle$-translates of $[m, gm]$ form axis. In fact since $o \in [m, gm]$ and $go \in [gm, g^2m]$ we need only show that $d(o, go) = 2d(o, gm)$.

$$d(o, go) = d(gv, g^2v) - 2d(o, gv) = d(v, gv) - 2\left(\frac{1}{2}d(v, gv) - d(o, gm)\right) = 2d(o, gm). \qquad \blacksquare$$

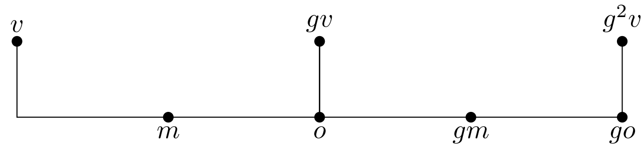


Figure II.1

There are four possibilities for a group G acting on a G-tree T which has no G-invariant proper subtrees [23, Section 2.2], [3, Remark 4]:

(E) *Elliptic*: T is a point and all elements act elliptically.

(H) *Hyperbolic*: There are two elements in G which act hyperbolically, whose axes intersect in a compact set. Then large powers of these elements generate a subgroup isomorphic to F_2.

(P) *Parabolic*: There is an infinite ray (half-infinite path) R in T such that $gR \cap R$ is an infinite ray for all $g \in G$. Then G fixes an *end* of T (q.v. next section), i.e. a point at infinity.

(D) *Dihedral*: T is a line and G acts via a surjective homomorphism $G \mapsto D_\infty$.

1C Ends of Groups and Stallings' Theorem

In this section we shall consider Stallings' Theorem characterising when finitely generated groups split as a graph of groups with finite edge groups.

Definition II.6. Let X be a topological space, $K_1 \subset K_2 \subset K_3 \subset \cdots$ an ascending sequence of compact subsets such that $\bigcup_i K_i = X$. The number of **ends** of X, $e(X)$ is the limit as $i \to \infty$ of the number of connected components of $X - K_i$

Intuitively $e(X)$ is the number of connected components of X "at infinity". The spaces $\mathbb{R}^0$, $\mathbb{R}$, and $\mathbb{R}^2$ have 0, 2, and 1 end respectively.

Definition II.7. Two metric spaces M_1 and M_2 are **quasi-isometric** if there is a function $f : M_1 \mapsto M_2$ and non-negative constants A, B, and C such that:

1. For all $x, y \in M_1$
$$\frac{1}{A}d_1(x, y) - B \leq d_2(f(x), f(y)) \leq Ad_1(x, y) + B$$
2. For all $y \in M_2$, there is $x \in M_1$ such that $d_2(f(x), y) \leq C$.

Being quasi-isometric is an equivalence relation which only cares about the coarse structure of the spaces M_1 and M_2. If M_1 and M_2 are quasi-isometric then $e(M_1) = e(M_2)$. If G is a finitely generated group, and S and S' are finite generating sets, then the Cayley graphs $\Gamma(G, S)$ and $\Gamma(G, S')$ are quasi-isometric, hence we can make the following definition.

Definition II.8. Let G be a finitely generated group with finite generating set S. The number of **ends** of G, $e(G)$ is equal to $e(\Gamma(G, S))$.

Remark II.9. It is easy to show in fact that if K is a connected simplicial complex on which G acts freely with finite quotient, then K is quasi-isometric to $\Gamma(G, S)$, and hence $e(K) = e(G)$ is independent on the choice of K.

It is clear that G is finite if and only if $e(G) = 0$. More generally we have the following.

Lemma II.10. *[19, Corollary 5.9] Let G be finitely generated, then $e(G)$ is 0, 1, 2, or ∞.*

Proof. Assume $e(G)$ is not 0, 1, or ∞, and write $n = e(G)$. Let S be a finite generating set of G, and let K be a connected compact subgraph of the Cayley graph $\Gamma(G, S)$ which separates the n ends of $\Gamma(G, S)$ i.e. $\Gamma(G, S)\backslash K$ has n connected components which have non-compact closure. Such a K clearly exists by the definition of ends. Choose an element $g \in G$ such that $gK \cap K = \emptyset$, g exists because K is compact. Removing gK also separates the n ends since $\Gamma(G, S)$ is homogeneous, and K must be contained in one of these ends. Similarly gK must be contained in one of the ends after removing K. Removing a compact connected set which contains K and gK produces $2(n - 1)$ ends, and the only solution to $2(n - 1) = n$ is to take $n = 2$. ■

The following characterisation of groups with two ends will be useful later.

Theorem II.11. *[7, Theorem 9.22] Let G be finitely generated, $e(G) = 2$ if and only if G is virtually infinite cyclic, i.e. it has a subgroup of finite index isomorphic to $\mathbb{Z}$.*

We can now state Stallings' Theorem which is the motivation for the next chapter. An "elementary" proof can be found in, for example, [12]. We shall prove Stallings' theorem for finitely presented groups in Section III.2.

Theorem II.12 (Stallings' Theorem)**.** *[22, 21] A finitely generated group G splits non-trivially over a finite subgroup if and only if $e(G) > 1$.*

II.2 Complexity of Group Splittings

When studying how groups split as graphs of groups, it is natural to try and associate the "complexity" of the group with the complexity of its decompositions, and this is what *accessibility* is all about. In order to do this we need to first rule out ways of producing complicated graphs of groups which correspond to trivial splittings. For example any group splits as an amalgam over itself $G = G *_G G$ so we make the following definitions.

2A Preliminary Definitions

Definition II.13. We say a graph of groups decomposition $\mathcal{G}$ of a group G is **reduced** if all vertex groups properly contain the edge groups incident to them as subgroups except possibly when the edge is a loop. If $\mathcal{G}$ is not reduced, then we can easily obtain a reduced graph of groups decomposition of G by recursively collapsing vertices across edges which do not satisfy the above condition (see for example Figure IV.2 on page 21).

A nice property of reduced graphs of groups is the following lemma which says that no two vertices have the same label up to conjugacy (the same is not true for edges).

Lemma II.14. *[16, Lemma 3] Let $\mathcal{G}$ be a reduced graph of groups and let V and U be vertices. If $gVg^{-1} \subset U$ then $U = V$ and $g \in V$.*

Proof. If $V \neq U$, or $U = V$ but $g \notin V$ then V stabilises the distinct vertices v and $g^{-1}u$ in the Bass-Serre tree $\tilde{\Gamma}$. Hence V stabilises the geodesic path between these two vertices, and in particular an edge e incident to v. Hence in $\mathcal{G}$ we have $V = E$ contradicting reducedness. ■

If we have any graph of groups decomposition of G then we can collapse any connected subgraph (i.e. replace the subgraph with a vertex labelled by the fundamental group of the subgraph) to get a simpler graph of groups decomposition of G. It is useful to consider the reverse of this process.

Proposition II.15. *[16, Section 2] Let $\mathcal{G}$ be a graph of groups decomposition of G, and let H be a subgroup of G labelling a vertex of $\mathcal{G}$. Let $\mathcal{H}$ be a graph of groups decomposition of H. If every edge group of $\mathcal{G}$ incident to H is an H-conjugate of a subgroup of some vertex group of $\mathcal{H}$, then we can replace the vertex H in $\mathcal{G}$ by the graph $\mathcal{H}$, attached by appropriate edges, to get a more complicated graph of groups decomposition $\mathcal{G}'$ of G. If $\mathcal{H}$ satisfies the above conditions we call it* ***compatible****.*

It is clear that if we start off with some complicated graph of groups $\mathcal{G}'$ and collapse some connected sub-graph of groups decomposition $\mathcal{H}$ to get a graph of groups $\mathcal{G}$, then $\mathcal{H}$ is compatible with $\mathcal{G}$. Hence we can construct any finite graph of groups decomposition of G by starting off with the single vertex labelled G (the trivial graph of groups) and iteratively split vertex groups as amalgamated products or HNN extensions, replacing the vertex with either a segment or a loop as appropriate and reducing. This leads to the notion of a splitting sequence, see Definition II.20.

Requiring reducedness is one way of avoiding producing complicated graph of groups decompositions which do not correspond to complicated algebraic splittings of the group. Another way is to demand that Γ is *minimal*.

Definition II.16. A G-tree T is **minimal** if it contains no proper G-invariant subtrees.

If we do not demand minimality then given a G-tree we can equivariantly attach as many other G-trees as we like to produce arbitrarily complicated graph of groups decompositions.

Remark II.17. If G is finitely generated and $\mathcal{G}$ is a graph of groups coming from a minimal G-tree, then the underlying graph of $\mathcal{G}$ is finite [1, Proposition 7.9].

2B Types of Accessibility

Accessibility is about bounding the complexity of graph of groups decompositions of a given group G. Conditions of reducedness and minimality are useful but in general are not sufficient, one must also put conditions on the splitting groups. Recall we say G **splits over** a subgroup H if there is a graph of groups decomposition of G which has H as an *edge group* (c.f. Definitions I.1 and I.2).

Definition II.18. A collection of groups $\mathcal{C}$ is a **class of groups** if the collection is closed under isomorphism, i.e. $G \in \mathcal{C}$ and $H \cong G$ implies $H \in \mathcal{C}$. In particular a class of subgroups of a group G must be closed under conjugation (recall the labels of a graph of groups coming from a G-tree are only defined up to conjugation).

Example II.19. Here are three classes of groups which commonly arise in this context. The class of finite groups, the class of **slender** groups which are groups all of whose subgroups are finitely generated, and the class of **small** groups which are groups which do not contain F_2 as a subgroup (the reason for considering this class is to preclude the possibility of *hyperbolic* actions on trees, see (H) in Section 1B above). It is well-known that F_2 contains countably generated subgroups so is not slender. Hence we have the inclusions of classes

$$\{\text{finite}\} \subset \{\text{slender}\} \subset \{\text{small}\}.$$

We can define various notions of accessibility.

Definition II.20. Let G be a group, and $\mathcal{C}$ a class of groups. A **splitting sequence** for G over $\mathcal{C}$ is a sequence of distinct reduced graph of groups decompositions $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \dots$ where $\mathcal{G}_1$ is the trivial decomposition (consisting of a single vertex), and for each $i > 1$, $\mathcal{G}_i$ is obtained from $\mathcal{G}_{i-1}$ by compatibly splitting a vertex group of $\mathcal{G}_{i-1}$ as either a single amalgamated product or HNN extension with edge group in $\mathcal{C}$, followed by reducing.

A group G is **accessible**[2] over $\mathcal{C}$ if there is some number $N(G)$, depending only on G, which bounds the length of any splitting sequence of G over $\mathcal{C}$. G is said to be **hierarchical accessible**[3] over $\mathcal{C}$ if there is a bound (again only depending on G) on the length of any sequence of groups $G = V_1, V_2, V_3, \dots, V_n$ such that V_i is a vertex group in a non-trivial graph of groups decomposition of V_{i-1} over $\mathcal{C}$.

[2] Historically a group was said to be *accessible* if it was accessible over finite groups, see for example [19, Section 7].
[3] Some authors refer to hierarchical accessibility as strong accessibility.

Example II.21. [3, Remark 4] Here is a very simple counter example to generalised accessibility. The free group on two generators F_2 admits arbitrarily long reduced graph of groups decompositions of the form:

$$\begin{tikzcd} \langle x\rangle \arrow[r, dash, "\langle x^2\rangle"] & \langle x^2, y^8\rangle \arrow[r, dash, "\langle x^4, y^8\rangle"] & \langle x^4, y^4\rangle \arrow[r, dash, "\langle x^8, y^4\rangle"] & \langle x^8, y^2\rangle \arrow[r, dash, "\langle y^2\rangle"] & \langle y\rangle \end{tikzcd}$$

This shows that F_2 is not accessible over its subgroups. More complicated counter examples have been constructed over finite [9] and small [2] subgroups.

It is clear that hierarchical accessibility implies accessibility, but in general the reverse is not true. A case when hierarchical accessibility is stronger than accessibility would be when we have a terminal graph of groups decomposition in a maximal splitting sequence for G over $\mathcal{C}$ in which one of the vertex groups splits non-trivially but *non-compatibly* over $\mathcal{C}$.

2C Some Results

The first result to do with accessibility (before it got that name) is Grushko's Theorem (1940) which concerns free products, i.e. amalgamated products over the class of trivial groups.

Theorem II.22 (Grushko's Theorem)**.** *Let $rk(G)$ denote the minimum number of generators of G.*

$$rk(G_1 * G_2) = rk(G_1) + rk(G_2)$$

Given any graph of groups decomposition $\mathcal{G}$ of a group G with trivial edge groups, then mapping the vertex groups to $\{1\}$ induces a surjection $G \mapsto F_{\beta_1}$ where β_1 is the first Betti number of the underlying graph of $\mathcal{G}$. Hence β_1 is bounded by $rk(G)$ [3, Remark 2], and so G can be decomposed as at most $rk(G)$ free products and HNN extensions. Hence finitely generated groups are hierarchical accessible over trivial edge groups.

As mentioned in Example II.21 finitely generated groups tend not to be accessible over larger classes of groups, however there are good results for finitely *presented* groups. In 1985 M.J. Dunwoody was able to prove accessibility of finitely presented groups over finite subgroups [10], we shall discuss this proof in detail in the next chapter. In 1991 M. Bestvina and M. Feighn proved accessibility of finitely presented groups over the class of small subgroups by a careful analysis of each of the cases in Section 1B using a technique called *folding* [3]. Most recently L. Louder and N. Touikan proved hierarchical accessibility of finitely presented groups over slender edge groups [13]. An important tool in this later work is the construction of so-called JSJ-decompositions.

Definition II.23. Let G be a group and $\mathcal{C}$ a class of groups. A graph of groups decomposition $\mathcal{G}$ of G over $\mathcal{C}$ is a **JSJ-decomposition** of G over $\mathcal{C}$ if whenever G splits as $A *_H B$ or $A*_{H,t}$ with $H \in \mathcal{C}$, there is some vertex group of $\mathcal{G}$ which contains a conjugate of H as a subgroup.

JSJ-decompositions capture in some sense all possible splittings of G over $\mathcal{C}$ in an essentially unique way. They were introduced by E. Rips and Z. Sela in [18], and constructed for finitely presented groups over slender subgroups by M.J. Dunwoody and M.E. Sageev [11].

In the final chapter we shall discuss the work of M.L. Mihalik and S. Tschantz who analysed the splitting theory of a subclass of finitely presented groups called Coxeter groups (q.v. Example II.2) [16]. For such groups, graph of groups decompositions can be constructed and studied easily just by looking at the group presentation, and we shall give a much shorter proof of the main theorem of the next chapter in the case of Coxeter groups. Stronger accessibility results and JSJ-decompositions of Coxeter groups have also been studied in a similar way [15, 17].

III Dunwoody Accessibility of Finitely Presented Groups

Stallings' Theorem forms the starting point of M.J. Dunwoody's proof of the accessibility of finitely presented groups over finite subgroups. In fact the result applies to *almost* finitely presented groups.

Definition III.1. A group G is **almost finitely presented** if there is a connected simplicial complex K with $H^1(K, \mathbb{Z}_2) = 0$, such that G acts on K freely and $G\backslash K$ is finite.

All finitely presented groups are almost finitely presented, as can be seen by considering the action of G on its Cayley 2–complex (see the paragraph after Example I.5), however the converse is not true. The relevance of the cohomological condition will be made concrete in Section III.1 below. In this chapter we shall prove the following accessibility theorem due to M.J. Dunwoody.

Theorem III.2. *[10, Theorem 5.1] Let G be an almost finitely presented group, then G is accessible over finite groups.*

For splittings over finite groups there are no compatibility issues, hence it is sufficient to show that there is a G-tree T with all edge stabilisers finite such that all vertex stabilisers have at most one end [19, Lemma 7.1]. The way this is done is broadly as follows. Consider a 2–complex K satisfying the conditions above and cut this up equivariantly along generalised essential closed curves (called *tracks*) which "separate ends". The dual graph to this complex with respect to these tracks is a G-tree, so a separation of the ends of K corresponds to a splitting of G. The final ingredient is to use the combinatorics of $G\backslash K$ to bound the number of essentially different tracks which separate K, and thus bound the number of edges in the associated graph of groups.

III.1 Tracks

We want to generalise the notion of simple closed curves in a surface to a simplicial 2–complex K. Throughout let $|K|$ denote the geometric realisation of K.

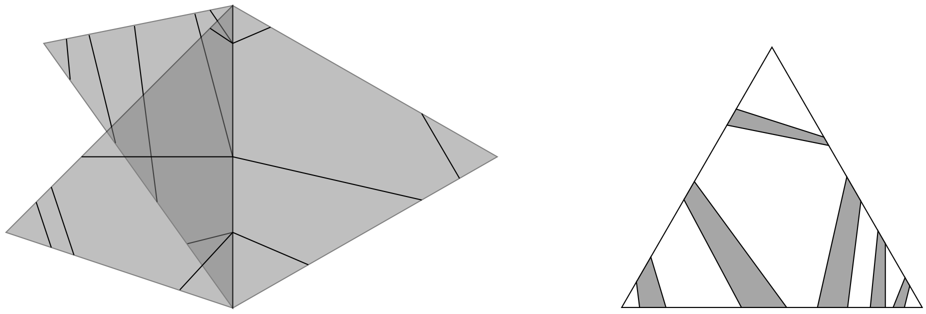
Definition III.3. Let K be a 2–complex, a **track** is a subset t of $|K|$ which satisfies the following:

1. t is connected,
2. For each 2–simplex σ of K, $t \cap |\sigma|$ is a disjoint union of finitely many straight lines joining distinct edges of $|\sigma|$, and
3. For γ a 1-simplex of K which is not a face of a 2–simplex, either $t \cap |\gamma| = \emptyset$ or t is a single point in the interior of $|\gamma|$.

It follows from this definition that if more than one 2–simplex meet in a face containing a point of a track t then each of those simplices contains a line segment belonging to t which meets that point, see Figure III.1a. As well as generalising closed curves we also want to generalise whether a regular neighbourhood of such a curve is an embedded annulus or Möbius band.

Definition III.4. Let K be a 2–complex, a **band** is a subset B of $|K|$ which satisfies the following:

1. B is connected,
2. For each 2–simplex σ of K, $B \cap |\sigma|$ is a disjoint union of finitely many components, each of which is bounded by two closed intervals in the interiors of distinct faces of $|\sigma|$ together with the two disjoint lines joining the end points of these intervals, and
3. For γ a 1–simplex of K which is not a face of a 2–simplex, either $B \cap \gamma = \emptyset$ or B is an interval in the interior of γ.



(a) An example of a track t restricted to three 2–simplices which meet in a single edge.

(b) An example of a band restricted to a 2–simplex.

Figure III.1

Given a band B, q.v. Figure III.1b, we can obtain a track $t(B)$ by joining the midpoints of the components of $B \cap |K^{(1)}|$ by straight lines in the appropriate 2–simplices, and conversely we can obtain a band from a track t by taking an ε-neighbourhood.

A band B is **untwisted** if it is homeomorphic to $t(B) \times [0,1]$, so ∂B has two boundary components each homeomorphic to $t(B)$. Otherwise we say that B is **twisted**, in which case ∂B double covers $t(B)$. Similarly we say a track t is **(un)twisted** if there is a band B which is (un)twisted such that $t(B) = t$. Two tracks t_1 and t_2 are **parallel** if there is an untwisted band B such that $\partial B = t_1 \cup t_2$.

We shall now explain the significance of the cohomological condition in the definition of almost finitely presented groups. It is clear that a twisted track cannot separate $|K|$, in addition we have the following.

Proposition III.5. *[10, Proposition 2.1] Let K be a 2–complex, $H^1(K, \mathbb{Z}_2) = 0$ if and only if K contains no twisted tracks. More generally suppose* $\operatorname{rank}(H^1(K, \mathbb{Z}_2)) = \beta$ *is finite and let $T = \{t_1, \ldots, t_n\}$ be a set of disjoint tracks. Then $|K| - \bigcup_i t_i$ has at least $n - \beta$ components.*

Proof. Let t be a track and define a 1–cochain $z(t)$ as follows: for a 1–simplex γ, $z(t)\gamma = \#(|\gamma| \cap t) \bmod 2$. For a 2–simplex σ, $\partial|\sigma| \cap t$ has an even number of points, hence $\delta z(t) = 0$ so $z(t)$ is a 1–cocyle. $z(t)$ is a coboundary if and only if there is $f \in H^0(K, \mathbb{Z}_2)$ such that $z(t)\gamma = (\delta f)\gamma = fv_1 + fv_2$ where γ is a 1–simplex with endpoints v_1 and v_2. $z(t)\gamma = 1$ if and only if locally v_1 and v_2 are on "opposite sides" of t, and $fv_1 + fv_2 = 1$ if and only if exactly one of fv_1 and fv_2 equals 1. Thus such an f exists if and only if the vertices of K can be "globally" partitioned into two sets, one on either side of t, i.e. if and only if t separates $|K|$, and hence is untwisted.

Thus $z(t)$ is an non-zero element of $H^1(K, \mathbb{Z}_2)$ if and only if t is twisted. Moreover since no non-empty disjoint union of twisted tracks separates $|K|$, the corresponding elements of $H^1(K, \mathbb{Z}_2)$ are linearly independent. The proposition is now clear. ■

We can use this proposition to bound the number of disjoint non-parallel tracks in a finite 2–complex. This is in analogy with bounding the number of disjoint closed curves in a surface, no two of which bound an annulus. This is the bound which is used to prove accessibility.

Theorem III.6. *[10, Theorem 2.2] Let L be a finite 2–complex and define $n(L) = 2\beta + v_L + f_L$ where $\beta = \operatorname{rank}(H^1(K, \mathbb{Z}_2))$, v_l is the number of vertices, and f_L is the number of 2–simplices. Suppose $t_1, \ldots, t_k$ are disjoint tracks in $|L|$, if $k > n(L)$ then there are indices $i \neq j$ such that t_i and t_j are parallel.*

Proof. Consider a 2–simplex σ of L, and let D be the closure of a component of $|\sigma| - \bigcup_i t_i$; D is a disc. We say D is *good* if $\partial D \cap \partial|\sigma|$ consists of two components in distinct faces of σ. If D is not good and there is just one component then its closure must contain a vertex of σ, otherwise D has three boundary components in $\partial|\sigma|$ and is "central" in $|\sigma|$, see Figure III.2.

If $k > n(L)$ then $|L| - \bigcup_i t_i$ has more than $v_L + f_L + \beta$ components by the previous proposition. At most $v_L + f_L$ of these components are unions of discs which are not *good*, and hence are discs themselves, leaving at least $\beta + 1$ bands. Since $|L|$ contains at most β twisted bands, at least one of these bands is untwisted, and its boundary consists of two tracks t_i and t_j which are parallel. ■

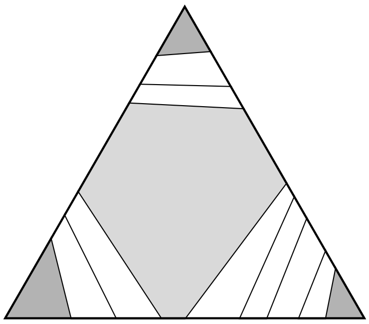


Figure III.2: A 2–simplex containing a collection of disjoint tracks. The shaded regions correspond to discs which are not good.

This next result is clear (by considering Figure III.1a or III.2).

Proposition III.7. *[10, Proposition 2.3] Let K be a 2–complex, and $J \subset |K^{(1)}| - |K^{(0)}|$ be a finite collection of points which satisfies the following. For each 2–simplex σ let $j_i \geq 0$ be the number of points of J in each edge of $|\sigma|$ for $i = 1, 2, 3$. We require $j_1 + j_2 + j_3 = 2m$ for some integer m (so that there is an even number), and $j_i \leq m$ for each i (so that no edge has more than half the points). Then there exists a unique set of disjoint tracks T in $|K|$ such that $|K^{(1)}| \cap \bigcup T = J$.*

The conditions on the numbers j_i are the obvious necessary conditions for such a set T to exist. If we have a collection of tracks which intersect, after possibly perturbing them so that the do not intersect in $|K^{(1)}|$ we can replace this collection with a collection of non-intersecting tracks.

III.2 Minimal Tracks

From now on we shall assume that we have a connected 2–complex K with $H^1(K, \mathbb{Z}_2) = 0$, in particular it contains no twisted tracks, or in other words, all tracks separate $|K|$. For our purposes we do not want to consider arbitrary tracks. Thinking back to the motivation in terms of graphs of spaces, suppose two of those spaces are joined by a cylinder. If we choose a simple closed curve in the cylinder which separates these two spaces, this corresponds to a splitting of the fundamental group over $\mathbb{Z}$. In particular we need to choose an *essential* simple closed curve, one that does not bound a disc. Also we want to exclude unnecessarily complicated closed curves which maybe double back on themselves, or wrap around several times. We generalise this to tracks by only considering *minimal* tracks.

Definition III.8. For a track t in $|K|$ let $||t|| = \#(|K^{(1)}| \cap t)$. Such a track is minimal if:

1. $||t||$ is finite,
2. The two components of $|K| - t$ each contain infinitely many vertices, and
3. If t' is another track satisfying the above, then $||t'|| \geq ||t||$.

When a minimal track t exists we set $m(K) = ||t||$.

Remark III.9. If t is a minimal track which meets a 2–simplex $|\sigma|$ in $|K|$ then their intersection consists of a single line segment. t can be thought of as a topological graph (with vertices $t \cap |K^{(1)}|$), and a band associated to t, which by our hypothesis is untwisted, can be "thickened" to be the union of all 2–simplices containing a point of t. This subcomplex is homeomorphic to the product of the graph t, and the interval $[0, 1]$ which we shall call the **finite piece** associated to t (this name follows the definition of *slender pieces* in [11]).

If U is a set of vertices of $|K|$, set $U^* = |K^{(0)}| - U$ to be its complementary set of vertices and let $w(U)$ be the number (possibly infinite) of edges of $|K|$ which have one vertex in U and the other in U^* (we say such an edge **joins** U and U^*). The next result characterises the existence of a minimal track based on the obvious conditions on U, U^* and $w(U)$.

Proposition III.10. *[10, Proposition 3.1] $|K|$ contains a minimal track t if and only if there is a set of vertices of $|K|$, U, such that U and U^* are infinite, but $w(U)$ is finite.*

Proof. The forward direction is clear, for the converse, let $J \subset |K^{(1)}| - |K^{(0)}|$ be a set of points obtained by choosing a single point in each edge joining U and U^*. J satisfies the conditions of Proposition III.7 so we get a disjoint collection of tracks T, with $||t||$ finite for each $t \in T$. If all of the tracks failed the second condition in the above definition it is easy to see that necessarily one of U or U^* is finite, a contradiction. Hence one of the tracks in T satisfies the second condition, so there must exist a minimal track. ■

If we have U as in the proposition so that there exist minimal tracks, then it is clear that $m(K) \leq w(U)$. Moreover if we happen to have equality then T as constructed in the proof must consist of a single minimal track. Compare this theorem to the definition of ends (Definition II.6), and it is clear that K has more than one end if and only if there exist minimal tracks in $|K|$.

Theorem III.11. *[10, Theorem 3.3] Let $S = \{t_1, \ldots, t_n\}$ be a set of not necessarily disjoint minimal tracks in $|K|$, such that no two intersect in $|K^{(1)}|$. If $J = |K^{(1)}| \cap (\bigcup S)$ and T is the set of disjoint tracks coming from Proposition III.7, then T consists of n disjoint minimal tracks.*

Proof. Consider first the case $n = 2$, if $t_1$and t_2 are disjoint we are finished, so assume they intersect. We must have that T consists of at most two tracks, otherwise one of the tracks $t' \in T$ would have $||t'|| < ||t_1|| = ||t_2||$, contradicting minimality. For similar reasons, if T contains exactly two tracks then both the resulting tracks must be minimal. Hence we need to rule out the possibility that $T = \{t\}$ with $||t|| = \#J = 2m(K)$. If t_1 and t_2 have the same associated finite piece (q.v. Remark III.9), then T must contain two tracks because this finite piece is untwisted (in this case t_1 and t_2 are just "perturbations" of one another, in particular the tracks in T are parallel). Hence we are safe to assume that t_1 and t_2 do not have the same finite pieces.

Let $|K^{(0)}| = U \sqcup U^*$ be the partition of the vertices of $|K|$ associated to t_1, and similarly let $|K^{(0)}| = V \sqcup V^*$ be associated to t_2, and assume we have chosen to label these sets such that $U \cap V$ and $U^* \cap V^*$ are both infinite. Let $|\sigma|$ be a 2–simplex in which t_1 and t_2 intersect, then the picture will be as in Figure III.3. The dashed lines represent the segments in $\bigcup T$ which replace those which intersect. By our assumptions there is at least one 2–simplex which looks like Figure III.3b and hence there is a component of $\bigcup T$ which separates $U \cap V$ from $(U \cap V)^*$, and a component which separates $U^* \cap V^*$ from $(U^* \cap V^*)^*$. Moreover these two pairs of sets of vertices are distinct, hence T must contain two distinct components.

The case for general n follows by induction on $N = \sum_{i,j} \#(t_i \cap t_j)$, write $N_{ij} = \#(t_i \cap t_j)$. If $N = 0$ the tracks are already disjoint so there is nothing to prove. Assume $N > 0$, and choose indices $p < q$ such that $N_{pq} > 0$. By the first part of the proof there are disjoint minimal tracks t'_p and t'_q such that $(t'_p \cup t'_q) \cap |K^{(1)}| = (t_p \cup t_q) \cap |K^{(1)}|$. Set $t'_i = t_i$ if $i \notin \{p, q\}$ and let $S' = \{t'_1, \ldots, t'_n\}$, $N'_{ij} = \#(t'_i \cap t'_j)$ and $N' = \sum_{ij} N'_{ij}$. We claim $N' < N$ which completes the proof since $J = |K^{(1)}| \cap (\bigcup S')$. Indeed it is clear that $N'_{pq} = 0$; let $i \notin \{p, q\}$ be an index and suppose $t_i \cap (t'_p \cup t'_q)$ is a single point in some 2–simplex $|\sigma|$, then t_i must intersect t_p or t_q in $|\sigma|$. If $t_i \cap (t'_p \cup t'_q)$ consists of two points then t_i must intersect both t_p and t_q in $|\sigma|$, see Figure III.3. Hence $N'_{ip} + N'_{iq} \leq N_{ip} + N_{iq}$ and $N' < N$. ■

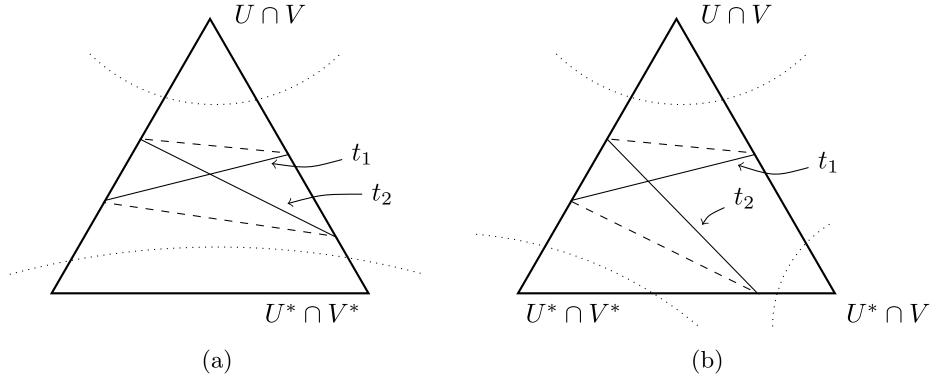


Figure III.3: The two possibilities for a 2–simplex in which t_1 and t_2 intersect. The segments of these tracks are replaced by the dashed segments in $\bigcup T$, and the separated sets of vertices are indicated by dotted lines.

Minimality is crucial to ruling out the possibility that $T = \{t\}$ consists of a single "fused" track; non-minimal counter-examples are very easy to write down. Now let G be a group acting freely on K. The final ingredient before we can prove accessibility is to construct a disjoint G-equivariant set of minimal tracks.

Theorem III.12. *[10, Theorem 4.1] If $|K|$ contains a minimal track t, then there is a minimal track t' such that Gt' (the G-orbit of t') is a set of disjoint minimal tracks.*

Proof. For each $g \in G$ let t_g be a minimal track which is an ε–perturbation of gt so that no two tracks t_{g_1} and t_{g_2} intersect in $|K^{(1)}|$. If we let $J = |K^{(1)}| \cap \bigcup\{t_g\}_{g\in G}$ then $\#(J \cap |\gamma|) = \#(J \cap g|\gamma|) < \infty$ for all $g \in G$ and γ an edge of K, hence we can choose the t_g's so that J is G-equivariant. We can now apply Proposition III.7 to get a G-equivariant set of disjoint tracks T. We want to conclude that these are minimal, but in general there are infinitely many so we cannot immediately apply Theorem III.11.

Let $t \in T$ be a track, we shall show it is minimal. Since $||t||$ is finite, the finite piece of t must be locally finite. More generally there is a locally finite subcomplex K_f of K such that $\bigcup T \subset |K_f|$. Let K_0 be the *finite* subcomplex of K_f consisting of all simplices with a vertex a (combinatorial) distance at most $m(K)$ from a point of t. If a minimal track t' intersects the finite piece of t, then $t' \subset |K_0|$. Let A_0 be the finite set of tracks t_g such that $t_g \cap |K_0| \neq 0$ and set $J_0 = |K^{(1)}| \cap \bigcup A_0$. We can apply Theorem III.11 J_0 to get a set T_0 of disjoint minimal tracks.

Note that $J_0 = J \cap |K_0^{(1)}|$ so any component of T which intersects the finite piece of t is a component of T_0 which is minimal; in particular t is minimal, whence the theorem. ■

As a brief aside, we can use the above theorem to prove one direction of Stallings' theorem (Theorem II.12) for almost finitely presented groups: if such a group has more than one end, it splits over a finite subgroup.

Proof of Stallings' Theorem. Let G be an almost finitely presented group with more than one end, and let K be a connected 2-complex on which G acts freely with finite quotient. Then by Remark II.9, $e(K) > 1$ and so $|K|$ contains a minimal track by Proposition III.10. We can choose K such that $H^1(K, \mathbb{Z}_2) = 0$ and so applying Theorem III.12 we can find an equivariant set of disjoint minimal tracks T in $|K|$ all of which are separating. Consider the graph Γ whose vertex set is the set of connected components of $|K| - \bigcup_{t\in T} t$, and whose edge set is T. For $t' \in T$, the two components of $|K| - \bigcup_{t\in T} t$ whose closures contain t' are the endpoints of the edge t'. The connectedness of K implies Γ is connected, and since all tracks are separating, removing any edge separates Γ so Γ is a tree, and it is clear that G acts on Γ. One can see that $G\backslash\Gamma$ contains a single edge labelled with the stabiliser of some $t' \in T$, and since $||t'|| < \infty$ and the action of G on K is free, the stabiliser is finite. Thus $G\backslash\Gamma$ is a non-trivial graph of groups corresponding to a splitting of G over a finite subgroup. ■

Remark III.13. The converse can also be proved using ideas about tracks. If G splits over a finite subgroup, it has a non-trivial graph of groups $\mathcal{G}$ with a single edge labelled by the finite subgroup. Build the corresponding graph of spaces X using the presentation 2-complexes as in Section I.1. Let $\tilde{X}$ be the universal cover of X, on which G acts freely with finite quotient X, this can be mapped equivariantly onto the Bass-Serre tree of $\mathcal{G}$. The preimages of the midpoints of edges give rise to a disjoint set of minimal tracks, each of which separate ends of $\tilde{X}$, so in particular, $e(G) = e(\tilde{X}) > 1$. Some details are discussed in [11, Lemma 2.2 and Example 3].

III.3 Proof of the Theorem

Recall that we are trying to exhibit a G-tree for G almost finitely presented in which all edge stabilisers are finite, and all vertex stabilisers have at most one end. Given an almost finitely presented group G there is a 2–complex K on which it acts freely which contains no twisted bands. The idea is to cut up K equivariantly and consider the dual graph. The edges correspond to minimal tracks, so their stabilisers are finite, the vertices are obtained by cutting off the ends of $|K|$. We want to show (using Theorem III.6) that we can only cut up $|K|$ a finite number of times before we necessarily end up with connected components which have at most one end (in which case there will no longer be minimal tracks). For this we shall use the final, as yet unused, condition for a group G to be almost finitely presented, that $G\backslash K$ is finite.

Proof of Theorem III.2. If $e(G) \leq 1$ there is nothing to do, hence assume $e(G) > 1$. As in the previous proof, G acts freely on a connected simplicial complex K with $H^1(K, \mathbb{Z}_2) = 0$, and the quotient $L = G\backslash K$ is a finite simplicial complex. We can find an equivariant set of disjoint minimal tracks $T = Gt_1$ for t_1 some minimal track, and we consider the graph Γ whose vertex set is the set of connected components of $|K| - \bigcup_{g \in G} gt_1$, and whose edge set is T. As before, Γ is a G-tree. If all vertex stabilisers have at most one end then we are done, otherwise there is some vertex v of Γ whose stabiliser G_v has more than one end.

Let C be the component of $|K| - \bigcup_{g \in G} gt_1$ corresponding to v, and let $|K_v|$ be the subcomplex of $|K|$ consisting of all simplices (and their faces) which meet C, see Figure III.4. It is clear that G_v acts on K_v freely, so $e(|K_v|) > 1$; set $L_v = G_v\backslash K_v$, we would like L_v to be finite. This follows from the easy fact that $i_v : L_v \mapsto L$ induced by the inclusion $K_v \hookrightarrow K$ is injective on maximal simplices, together with the assumption that L is finite. This means we can repeat the above, trying to find a splitting of G_v by finding a minimal track t_2 in $|K_v|$. The only thing we want to avoid is t_2 intersecting one of the tracks in the G-orbit of t_1 which lies in K_v.

All translates of t_1 in $|K_v|$ are homotopic to the boundary components of their respective finite pieces, and in particular each is homotopic to a boundary component of $|K_v|$. This means that firstly they are not minimal tracks in $|K_v|$, and secondly no minimal track in $|K_v|$ is parallel to a translate of t_1. It also follows that we can choose a minimal track in $|K_v|$ which does not intersect any translate of t_1. Applying Theorem III.12 to such a track we can find a minimal track t_2 in $|K_v|$ which does not intersect the translates of t_1 and such that $G_v t_2$ is an equivariant set of disjoint minimal tracks. This gives a compatible splitting of G_v.

The final step is to prove that we cannot repeat this process indefinitely. We have noted that t_1 and t_2 are not parallel in $|K|$, we claim their projections are also not parallel in L. Indeed under the projection $|K| \mapsto |L|$, the preimage of a band in $|L|$ must be a union of bands in $|K|$. Theorem III.6 now guarantees that we cannot have an arbitrary number of non-parallel tracks in L, whence the theorem. ■

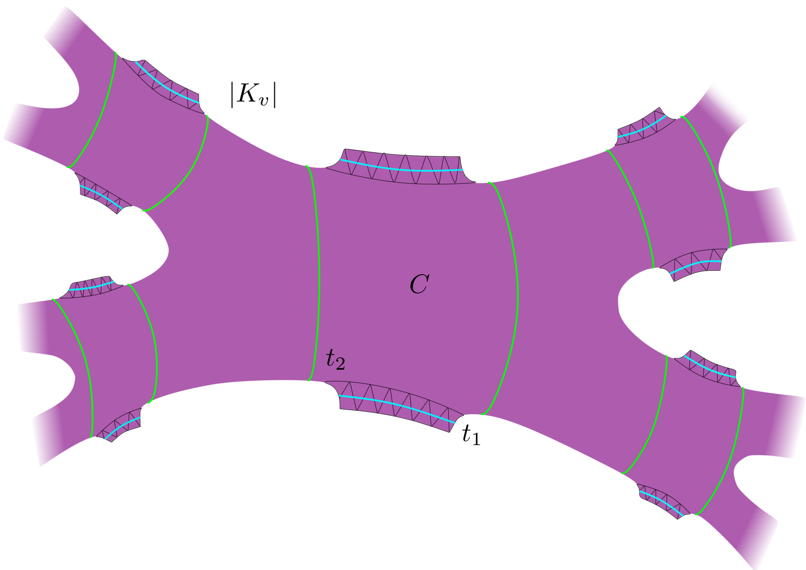


Figure III.4: An example of a component C of $|K| - \bigcup_{g \in G} gt_1$ extended slightly to give the subcomplex $|K_v|$, which has more than one end. t_1 and its translates are shown in blue, t_2 and its translates are shown in green.

The usefulness of tracks is not limited to this result, indeed one can see from the above proof that they are intimately related to splittings of groups. In [11] M.J. Dunwoody and M.E. Sageev use a method called track *zipping* in order to construct JSJ-decompositions of finitely presented groups over slender subgroups (recall definitions from Section II.2C). M.J. Dunwoody also used similar techniques to give an elementary proof of an equivariant sphere theorem, a version of Theorem III.12 one dimension higher (considering minimal essential spheres in a triangulated 3-manifold), [8]. Tracks, their properties, and their applications are discussed further in [6, Chapter VI].

IV Visual Decompositions of Coxeter Groups

In this final chapter we shall discuss the splittings of Coxeter groups and their accessibility. Because Coxeter groups are finitely presented, the results discussed in Section II.2C apply, and in particular M.J. Dunwoody's Accessibility Theorem. However Coxeter groups admit a particularly simple presentation and so the theory of their splittings is very nice. Coxeter groups possess a rich theory very closely tied to geometry, and they arise in many different areas of mathematics, making their study of independent interest to geometric group theorists. Interestingly however, the examples of Coxeter groups which are studied most often (spherical, Euclidean, and hyperbolic types), rarely split (they have the property (FA)). Consequently, for the class of Coxeter groups of most interest here, geometry plays very little role in what follows (at least in the formulation given here).

The splittings of Coxeter groups have been studied mostly by M. Mihalik and S. Tschantz, and this chapter is based on their paper [16]. We shall discuss so-called *visual* splittings, those which can be read off from the presentation. It turns out that all graph of groups decompositions of a Coxeter group are closely related in some sense to such a visual decomposition. We will also characterise the number of ends of a Coxeter group by looking at the presentation, and identify maximal (FA) subgroups. We shall end the chapter by reproving Dunwoody's accessibility result in the particular case of Coxeter groups.

IV.1 Coxeter Groups

For the general theory of Coxeter groups, consult [4] or [5].

Definition IV.1. A **Coxeter group** W is a group generated by reflections which acts discretely. Coxeter groups admit a presentation of the form

$$W = \langle S = \{s_1, \dots, s_k\} \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \ \forall\, 1 \leq i < j \leq k \rangle$$

for $m_{ij} \in \{2, 3, \dots, \infty\}$, where $m_{ij} = \infty$ means s_i and s_j are unrelated. We call the pair (W, S) a **Coxeter system**. If $S' \subset S$ is a subset of generators, the group $W_{S'} = \langle S' \rangle$ is again a Coxeter group with Coxeter system $(W_{S'}, S')$; such subgroups are called **visual**[1] **subgroups** of (W, S). A Coxeter system (W, S) is specified completely up to isometry by the numbers m_{ij}, and it is convenient to represent them diagrammatically.

Definition IV.2. Let (W, S) be a Coxeter system, its **presentation diagram** $\mathcal{V}(W, S)$ is a labelled graph with vertex set S, and an edge between s_i and s_j labelled by m_{ij} unless $m_{ij} = \infty$ in which case there is no edge (not to be confused with the Coxeter-Dynkin diagram which does not feature here).

Visual subgroups of (W, S) correspond to subgraphs of the presentation diagram of (W, S) which are induced by taking the span of a subset of vertices. If $\mathcal{V}'$ is an induced subgraph of $\mathcal{V}(W, S)$, then we shall denote the corresponding visual subgroup by $W_{\mathcal{V}'}$. We shall apply the adjective "visual" throughout this chapter to indicate properties and objects which can be visually read off from the presentation diagram. We shall see that almost all splitting properties of Coxeter systems are visual in this sense.

Example IV.3. Common examples of Coxeter groups come from the symmetry groups of regular polyhedra and periodic tilings. The symmetry group of the regular tetrahedron is the symmetric group S_4 generated by the transpositions $(1, 2)$, $(2, 3)$ and $(3, 4)$ with presentation diagram shown in Figure IV.1a.

As another example, consider the tiling of the hyperbolic plane shown in Figure I.4b on page 7, and let W be the group generated by the hyperbolic reflections in the three sides of the upper central triangle, and label these s_1, s_2, and s_3 going anti-clockwise starting from upper left side of the triangle. The pair

[1] This name appears to be due to M.L. Mihalik and S. Tschantz. Most of the literature uses the term *special* or *parabolic*.

Figure IV.1

of edges corresponding to s_1 and s_2 intersect at an angle of $\frac{\pi}{3}$ so $s_1 s_2$ is a hyperbolic rotation (elliptic isometry) by $\frac{2\pi}{3}$ and hence has order 3. The same is true for the pair s_2 and s_3. The edges corresponding to s_1 and s_3 are parallel and so $s_1 s_3$ is a hyperbolic translation (parabolic isometry) and so has infinite order. The presentation diagram for $(W, \{s_1, s_2, s_3\})$ is shown in Figure IV.1b.

The following result is due to J. Tits and the proof is by constructing a faithful representation of W as a reflection group in a real vector space. We state it as a fact.

Proposition IV.4. *[4, Corollary IV.1.3] Let (W, S) be a Coxeter system, the generators $s_i \in S$ represent distinct elements of W and the order of $s_i s_j$ in W is m_{ij}.*

IV.2 Visual Splittings

Before we consider general splittings of Coxeter groups, we shall first discuss splittings over visual subgroups where the factors are also visual.

Definition IV.5. Let (W, S) be a Coxeter system and $\mathcal{W}$ a graph of groups decomposition of W. $\mathcal{W}$ is a **visual graph of groups decomposition** of W if all vertex and edge groups are visual subgroups, and the homomorphisms of edge groups into vertex groups are given by inclusion.

Recall from Example II.2 that Coxeter groups do not split as HNN extensions, hence any graph of groups decomposition must be a tree of groups decomposition. It is clear that if a Coxeter system splits as a free product with visual factors, then these factors correspond to unions of the connected components of $\mathcal{V}(W, S)$. More generally from the definition of amalgamated products we have the following.

Proposition IV.6. *[16, Section 1] A Coxeter system (W, S) splits non-trivially and visually as an amalgamated product if and only if there are proper induced subgraphs $\mathcal{V}_1$ and $\mathcal{V}_2$ of the presentation diagram $\mathcal{V}(W, S)$ such that $\mathcal{V}(W, S) = \mathcal{V}_1 \cup \mathcal{V}_2$. In this case $W = W_{\mathcal{V}_1} *_{W_{\mathcal{V}_1 \cap \mathcal{V}_2}} W_{\mathcal{V}_2}$. We say that the subgraph $\mathcal{V}_1 \cap \mathcal{V}_2$* ***separates*** *$\mathcal{V}(W, S)$.*

Taking $\mathcal{V}_1 = \overset{s_1}{\bullet}\overset{3}{\text{———}}\overset{s_2}{\bullet}$ and $\mathcal{V}_2 = \overset{s_2}{\bullet}\overset{3}{\text{———}}\overset{s_3}{\bullet}$ in the previous example we have $\mathcal{V}_1 \cap \mathcal{V}_2 = \overset{s_2}{\bullet}$, giving the splitting $W = \langle s_1, s_2 \rangle *_{\langle s_2 \rangle} \langle s_2, s_3 \rangle$. This can be realised by a tree action by looking at Figure I.4b, W acts on the same tree as $SL_2(\mathbb{Z})$ giving a graph of groups with two segments which we can then reduce, see Figure IV.2.

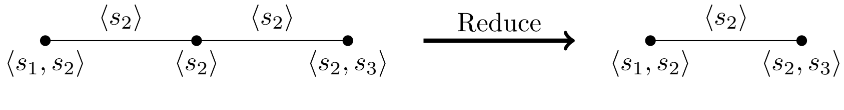


Figure IV.2

Definition IV.7. A visual subgroup $W_{S'}$ of a Coxeter group with Coxeter system (W, S) is called a **simplex subgroup** if its presentation diagram is a complete subgraph of $\mathcal{V}(W, S)$. $S' \subset S$ is called a **simplex**.

It is clear from Proposition IV.6 that the only visual graph of groups decomposition of a simplex subgroup is the trivial one, such subgroups might be called *visually* (FA) *subgroups* of W; *a priori* this is a weaker property that being (FA), but they are in fact equivalent for visual subgroups (q.v. Theorem IV.14). We can characterise visual decompositions as follows.

Lemma IV.8. *[16, Lemma 4] For a Coxeter system (W, S), a graph of groups $\mathcal{G}$ is a visual graph of groups decomposition for W if and only if the following are satisfied:*

1. *The underlying graph of $\mathcal{G}$ is a tree, all vertex and edge groups of $\mathcal{G}$ are visual subgroups of W, and the homomorphisms of edge groups into vertex groups are inclusions,*
2. *$\mathcal{V}(W, S)$ is the union of the presentation diagrams of the vertex groups of $\mathcal{G}$, and*
3. *For each generator $s \in S$, the subgraph of $\mathcal{G}$ consisting of all vertices and edges whose label contains s is connected.*

Proof. For the forward implication, 1 is clear. Suppose 2 did not hold and recall the definition of the fundamental group of a tree of groups. If a vertex of $\mathcal{V}(W, S)$ is not represented in the presentation diagrams of the vertex groups then $\pi_1(\mathcal{G})$ does not contain the corresponding generator so cannot be isomorphic to W (q.v. Proposition IV.4). Alternatively suppose all the vertices were represented but there is an edge which is not, then $\pi_1(\mathcal{G})$ is missing the corresponding relation so again cannot be isomorphic to W. In order to prove 3, recall by Proposition I.16 that we can lift the graph of $\mathcal{G}$ to a subtree of the Bass-Serre tree for $\mathcal{G}$, so that the labels are equal to the stabilisers. Let V and U be two vertex labels which contain s, and let v and u be their lifts. Then s stabilises v and u, and hence the unique geodesic path between them, which descends to a path in $\mathcal{G}$.

To prove the converse we need to show that if $\mathcal{G}$ satisfies 1–3, then $\pi_1(\mathcal{G})$ is isomorphic to W. By 2 every generator of W is in $\pi_1(\mathcal{G})$ and there are no extra generators from stable letters since the underlying graph is a tree. If a generator appears in more than one vertex group then 3 guarantees that they are identified in $\pi_1(\mathcal{G})$. Finally $\pi_1(\mathcal{G})$ has exactly the relations of W by the definition of the fundamental group and 2. Hence we have the required isomorphism. ∎

The main result of this section is the following theorem which says that any graph of groups decomposition of a Coxeter group W induces a visual graph of groups decomposition of W.

Theorem IV.9. *[16, Theorem 1] Let (W, S) be a Coxeter system and $\mathcal{G}$ a graph of groups such that W is a subgroup of $\pi_1(\mathcal{G})$. Then W has a visual graph of groups decomposition $\mathcal{W}$ such that vertex (resp. edge) groups of $\mathcal{W}$ are subgroups of conjugates of vertex (resp. edge) groups of $\mathcal{G}$. Moreover $\mathcal{W}$ can be chosen so that each visual subgroup of W which is a subgroup of a conjugate of a vertex group of $\mathcal{G}$ is a subgroup of a vertex group of $\mathcal{W}$.*

If $\mathcal{W}$ satisfies the full conclusion of the theorem then we say that it is a (W, S)-**visual decomposition coming from** $\mathcal{G}$.

Proof. Let $\tilde{\Gamma}$ be the Bass-Serre tree associated to $\mathcal{G}$, and construct a visual graph of groups decomposition $\mathcal{W}'$ of W with this (in general infinite) tree as its underlying graph. Recall from the end of Chapter I our notational conventions for the vertices and edges of $\tilde{\Gamma}$ and the groups of $\mathcal{G}$. For each vertex v (resp. edge e) of $\tilde{\Gamma}$ take the vertex (resp. edge) group of $\mathcal{W}'$ to be the visual subgroup of W generated by those $s \in S$ which stabilise v (resp. e). It is clear that this is a subgroup of a conjugate of V (resp. E). If s stabilises an edge, since the action of $\pi_1(\mathcal{G})$ is without inversion, then s stabilises the end points of that edge so we can take the homomorphisms to be inclusion. Each generator $s \in S$ has order 2 and so generates a finite group which in particular has the property (FA) so s fixes a vertex of $\tilde{\Gamma}$. If s_i and s_j are distinct generators in S which are connected by an edge in $\mathcal{V}(W, S)$, then the order of $s_i s_j$, m_{ij}, is finite and they generate a finite subgroup of W (the dihedral group of order $2m_{ij}$), thus this subgroup fixes a vertex. Hence every vertex and edge of $\mathcal{V}(W, S)$ is represented by a vertex or an edge in the presentation diagram of a vertex group. Finally if s stabilises two distinct vertices then it must stabilise the geodesic path between them, and so the subgraph of $\tilde{\Gamma}$ stabilised by s is a tree. It now follows from Lemma IV.8 that $\mathcal{W}'$ is a visual graph of groups decomposition of W.

We have established the first part of the theorem. For the moreover clause, note that if $W_{S'}$ is a visual subgroup of a conjugate of a vertex group of $\mathcal{G}$ then it stabilises a vertex of $\tilde{\Gamma}$ and so will be a subgroup of a vertex group of $\mathcal{W}'$. This proves the theorem, however the graph of groups we have constructed is in general infinite, but since W is finitely generated, it will not be minimal. However there is a finite set of vertices of $\tilde{\Gamma}$ whose stabilisers between them represent all vertices and edges of $\mathcal{V}(W, S)$, and even all visual subgroups mentioned in the moreover clause. Taking the subtree of $\tilde{\Gamma}$ spanned by these vertices gives a finite visual graph of groups decomposition $\mathcal{W}$ of W satisfying the theorem, and this can be reduced to give a reduced decomposition satisfying the theorem. ∎

IV.3 Ends and (FA) Subgroups

Recall Stallings' Theorem characterising the splittings of groups over finite subgroups. We can use this to visually characterise when a Coxeter group has more than one end. Recall that $e(W)$ is independent of the choice of Coxeter system.

Proposition IV.10. *[16, Corollary 16] Let W be a Coxeter group with Coxeter system (W, S), then the following are equivalent:*

1. *$e(W) > 1$,*
2. *W decomposes as a non-trivial visual amalgamated product $W_{S_1} *_{W_{S_3}} W_{S_2}$ for $S_i \subset S$ for each i and with W_{S_3} finite, and*
3. *$\mathcal{V}(W, S)$ contains a separating subgraph which is the a presentation diagram of a finite Coxeter group.*

Proof. $1 \Rightarrow 2$: Stallings' Theorem says that W splits non-trivially over a finite subgroup, and this must be as an amalgamated product $A *_C B$ since W admits no decompositions as an HNN extension. Apply Theorem IV.9 to this splitting to get a visual decomposition of W as an amalgamated product $W_{S_1} *_{W_{S_3}} W_{S_2}$. W_{S_3} is a subgroup of a conjugate of C and so must be finite, and W_{S_1} and W_{S_2} are subgroups of conjugates of A and B, in particular they are not the whole of W so this decomposition is non-trivial. $2 \Rightarrow 1$ follows immediately from Stallings' Theorem. $2 \Leftrightarrow 3$ is a restatement of Proposition IV.6. ■

The finite, or equivalently 0–ended, Coxeter groups have been classified in terms of their presentation diagram [4, Theorem VI.4.1], so in order to visually determine $e(W)$ we just need to be able to distinguish the 2–ended and ∞–ended cases from the presentation diagram. This is done using Theorem II.11.

Proposition IV.11. *[16, Corollary 17] A Coxeter group W with Coxeter system (W, S) is 2–ended if and only if $\mathcal{V}(W, S)$ contains a separating subdiagram $\mathcal{V}_0$ such that $W_{\mathcal{V}_0}$ is finite, $\mathcal{V}(W, S) - \mathcal{V}_0$ consists of two vertices, each of which is connected to every vertex of $\mathcal{V}_0$ be an edge labelled 2.*

If the second half of this proposition is satisfied, the two vertices which are separated generate a subgroup isomorphic to D_∞ of finite index, which in turn contains a copy of $\mathbb{Z}$ of finite index.

We want to say something about (FA) subgroups of Coxeter groups. We make the following definitions.

Definition IV.12. Let (W, S) be a Coxeter system and S' a subset of S. The **link** of S', $lk(S')$, is the subset of S joined to a vertex of S' in $\mathcal{V}(W, S)$. The **star** of S' is $st(S') = S' \cup lk(S')$.

Proposition IV.13. *[16, Lemma 25] Let (W, S) be a Coxeter system and let G be an (FA) subgroup of W, then G is a subgroup of a conjugate of a simplex subgroup of (W, S).*

Proof. If $\mathcal{V}(W, S)$ is a complete graph we are done, so assume not, and let $s, s' \in S$ be unrelated. Then $lk(s)$ separates s and s' so $W = W_{st(s)} *_{W_{lk(s)}} W_{S-\{s\}}$. Since G is an (FA) subgroup, it must be a subgroup of a conjugate of $W_{st(s)}$ or $W_{S-\{s\}}$ (q.v. Property 1, p. 9). Keep splitting until the lemma is realised, this will happen after at most $\#S < \infty$ splittings. ■

Theorem IV.14. *[16, Theorem 26] The maximal (FA) subgroups of a Coxeter group W are the conjugates of the visual subgroups of W generated by maximal simplices in $\mathcal{V}(W, S)$ for some (and equivalently any) Coxeter system (W, S) of W.*

Sketch Proof. Simplex subgroups have the property (FA) because if they split non-trivially, then by Theorem IV.9 they would split non-trivially and visually which we know they do not. Hence also their conjugates have the property (FA). By Lemma IV.13 we need only worry about simplex subgroups. The theorem follows from the easy fact which we leave as an exercise: if A is a subgroup generated by a maximal simplex which is contained in wBw^{-1} for B another simplex subgroup, then $A = B$ and $w \in B$ [16, Corollary 13]. ■

Since these results are independent of the choice of Coxeter system for (W, S), they have applications to the isomorphism problem of Coxeter groups, which is discussed in [16, Section 6–7].

IV.4 Accessibility of Coxeter Groups

We finish the chapter by reproving Dunwoody's accessibility result for Coxeter groups. Unlike Dunwoody's proof, the bound used to prove accessibility over finite edge groups is completely algebraic. Roughly speaking we associate a "density" to a graph of groups decomposition of a Coxeter group by counting the number of certain finite and visual subgroups of vertex groups, and show that decompositions cannot become arbitrarily diffuse. The key step is showing that this density strictly decreases if we non-trivially and compatibly split a vertex group, for which the following lemma is essential.

Lemma IV.15. *[16, Lemma 20] Let (W, S) be a Coxeter system and let $\mathscr{W}$ be a graph of groups decomposition of W with finite edge groups. Suppose a vertex group of $\mathscr{W}$ splits non-trivially as $A *_C B$ with C finite. Then there is a visual subgroup or a subgroup of a finite visual subgroup of W which is contained in a conjugate of B but not in any conjugate of A (and with A and B swapped).*

Proof. Let $\mathscr{W}'$ be the graph of groups obtained by replacing the split vertex in $\mathscr{W}$ by a segment corresponding to $A *_C B$ (q.v. Definition II.15). Let $\mathscr{W}'_{\text{vis}}$ be a (W, S)-visual graph of groups decomposition coming from $\mathscr{W}'$ as in Theorem IV.9, and let $\tilde{\Gamma}$ be the Bass-Serre tree of $\mathscr{W}'_{\text{vis}}$. The intersection of any conjugates of A and B, or of distinct conjugates of B, is contained in an edge group (since this intersection stabilises a geodesic path in $\tilde{\Gamma}$). Hence any such intersection must be finite so if any infinite vertex group of $\mathscr{W}'_{\text{vis}}$ lies in a conjugate of B, it cannot also lie in a conjugate of A and we are done. So now assume that there is no infinite vertex group lying in a conjugate of B; from the action of B on $\tilde{\Gamma}$ we get a reduced graph of groups decomposition of B with vertex and edge groups contained in conjugates of vertex and edge groups of $\mathscr{W}'_{\text{vis}}$, in particular all edge groups are finite. We claim that all vertex groups are finite as well. If not, let B_1 be an infinite vertex group, then it is contained in a conjugate of an infinite vertex group of $\mathscr{W}'_{\text{vis}}$ which is in turn contained in a conjugate of a vertex group of $\mathscr{W}'$ other than B. This is impossible by the argument above.

Thus the vertex groups of the graph of groups decomposition of B are finite, and moreover they are conjugate to subgroups of finite special subgroups. Replace the vertex B in $\mathscr{W}'$ by this graph of groups, and after reducing if necessary, we have a new graph of groups decomposition $\mathscr{W}''$ in which A is adjacent to a finite subgroup B_1 of B via an edge labelled $C_1 (= C$ if no reduction took place) which is properly contained in B_1. Now $B_1 \leq B$ cannot be contained in a conjugate of A because otherwise B_1 would stabilise a path on the Bass-Serre tree of $\mathscr{W}''$ to a coset of A and thus stabilise a coset of C_1. Then B_1 would be contained in a conjugate of C_1 which has strictly fewer elements. ∎

Theorem IV.16. *[16, Theorem 21] Coxeter groups are accessible over finite groups.*

Proof. Let (W, S) be a Coxeter system. For $G \leq W$ any subgroup, let $n(G)$ be the number of visual subgroups or subgroups of finite visual subgroups which are contained in any conjugate of G. Then $1 \leq n(G) \leq n(W) < \infty$. For $\mathscr{W}$ a finite graph of groups decomposition of W, let $c(\mathscr{W}) = (c_{n(W)}, \ldots, c_2, c_1)$ where c_i is the number of vertex groups G of $\mathscr{W}$ with $n(G) = i$. Let $\prec$ be the lexicographical ordering of $n(W)$–tuples of non-negative integers which is a well-ordering. If $\mathscr{W}$ reduces to $\mathscr{W}'$, then no c_i increases, and some c_i must decrease. If a vertex group G of $\mathscr{W}$ splits as $A *_C B$ with C finite to produce a new decomposition $\mathscr{W}''$ then every subgroup of a conjugate of A or B is a subgroup of a conjugate of G. However by the previous lemma there is some visual subgroup, or subgroup of a finite visual subgroup, which is contained in a conjugate of B and so of G, but not A. Hence $n(A) < n(G)$, and similarly $n(B) < n(G)$. Thus $c(\mathscr{W}'') \prec c(\mathscr{W})$ since $c_{n(G)}$ decreases by 1, and the only other terms which change are $c_{n(A)}$ and $c_{n(B)}$ which are later in the tuples. Since $\prec$ is a well-ordering, there can be no infinite splitting sequence of graph of groups decompositions of W over finite subgroups, thus W is accessible. ∎

The proof of Lemma IV.15 shows how useful Theorem IV.9 is allowing one to pass to visual decompositions in order to study general graphs of groups; indeed all decompositions are in some sense *close* to visual decompositions [16, Theorem 2]. The results proved here form the basis for strong accessibility results [15] and JSJ-decompositions [17] for Coxeter groups. This proof of accessibility over finite groups is much shorter than Dunwoody's proof, and the bounds can in principle be calculated explicitly. Everything in this chapter with the exception of Section IV.3 (but including Propositions IV.10 and IV.13) immediately generalises to finitely presented groups G which have a presentation $\langle A \mid R \rangle$ satisfying: (a) taking any proper subset of A or R defines a non-isomorphic group (c.f. Proposition IV.4); (b) A consists of torsion elements (c.f. Lemma II.3); and (c) if $A' \subset A$ and there is $r \in R$ which contains a or a^{-1} for each $a \in A'$ and no other generators, then $\langle A' \rangle$ is a finite subgroup (c.f. proof of Theorem IV.9).

Bibliography

[1] H. Bass. "Covering theory for graphs of groups". In: *J Pure Appl Algebra* 89 (1993), pp. 3–47.

[2] M. Bestvina and M. Feighn. "A counter example to generalised accessibility". In: *Arboreal Group Theory*. Springer-Verlag New York, 1991, pp. 133–368.

[3] M. Bestvina and M. Feighn. "Bounding the complexity of simplicial group actions on trees". In: *Invent Math* 103.3 (1991), pp. 449–470.

[4] N. Bourbaki. *Lie groups and Lie algebras*. Trans. by Andrew Pressley. Elements of Mathematics. Springer-Verlag Berlin Heidelberg New York, 2002. Chap. IV–VI. ISBN: 3–540–42650–7.

[5] K. S. Brown. *Buildings*. Springer-Verlang New York inc., 1989. Chap. I–III. ISBN: 0–387–98624–3.

[6] W. Dicks and M.J. Dunwoody. *Groups acting on graphs*. Cambridge studies in advanced mathematics 17. Cambridge University Press, 1989. ISBN: 0–521–23033–0.

[7] C. Druţu and M. Kapovich. *Geometric group theory*. Vol. 63. AMS Colloquium Publications. Americal Mathematical Society, 2018. ISBN: 978–1–4704–1104–6.

[8] M.J. Dunwoody. "An equivariant sphere theorem". In: *B Lond Math Soc* 17.5 (1985), pp. 437–448.

[9] M.J. Dunwoody. "An inaccessible group". In: *Geometric Group Theory* 1 (1993), pp. 75–78.

[10] M.J. Dunwoody. "The accessibility of finitely presented groups". In: *Invent Math* 81 (1985), pp. 449–458.

[11] M.J. Dunwoody and M.E. Sageev. "JSJ-splittings for finitely presented groups over slender groups". In: *Invent Math* 135.1 (1999), pp. 25–44.

[12] B. Krön. "Cutting up graphs revisited - a short proof of Stallings' structure theorem". In: *Groups Complex Cryptol* 2 (2010), pp. 213–221.

[13] L. Louder and N. Touikan. "Strong accessibility for finitely presented groups". In: *Geometry and Topology* 21 (2017), pp. 1805–1835.

[14] G.A. Margulis. "On the decomposition of discrete subgroups into amalgams". In: *Sel Math Sov* 1.2 (1981), pp. 197–213.

[15] M.L. Mihalik and S. Tschantz. "Strong accessibility of Coxeter groups over minimal splittings". In: *J Pure Appl Algebra* 216.5 (2012), pp. 1102–1117.

[16] M.L. Mihalik and S. Tschantz. "Visual Decompositions of Coxeter Groups". In: *Groups Geom Dyn* 3 (2009), pp. 173–198.

[17] J. Ratcliffe and S. Tschantz. "JSJ decompositions of Coxeter groups over FA subgroups". In: *Topol P* 42 (2013), pp. 57–72.

[18] E. Rips and Z. Sella. "Cyclic splittings of finitely presented groups and the canonical JSJ decomposition". In: *Ann Math* 146 (1997), pp. 53–109.

[19] P. Scott and T. Wall. "Topological methods in group theory". In: *Homological Group Theory*. London Mathematical Society Lecture Noter Series 36. Cambridge University Press, 1979, pp. 137–204. ISBN: 978–0–5212–2729–2.

[20] J. Serre. *Trees*. Springer-Verlag Berlin Heidelberg, 1980. ISBN: 978–3–540–44237–0.

[21] J.R. Stallings. *Group theory and three-dimensional manifolds*. Yale Mathematical Monographs 4. Yale University Press, 1971. ISBN: 0–300–01397–3.

[22] J.R. Stallings. "On torsion-free groups with infinitely many ends". In: *Ann Math* 88.2 (1968), pp. 312–334.

[23] J. Tits. "A "theorem of Lie-Kolchin" for trees". In: *Contributions to algebra. A collection of papers dedicated to Ellis Kolchin*. Academic press, inc., 1977, pp. 377–388. ISBN: 978–0–12–080550–1.

[24] J. Tits. "Sur le groupe des automorphismes d'un arbre". In: *Essays on topology and related topics*. Springer-Verlag Berlin Heidelberg, 1970, pp. 188–211. ISBN: 978–3–642–49199–3.