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The Many Complexities of Coxeter Groups

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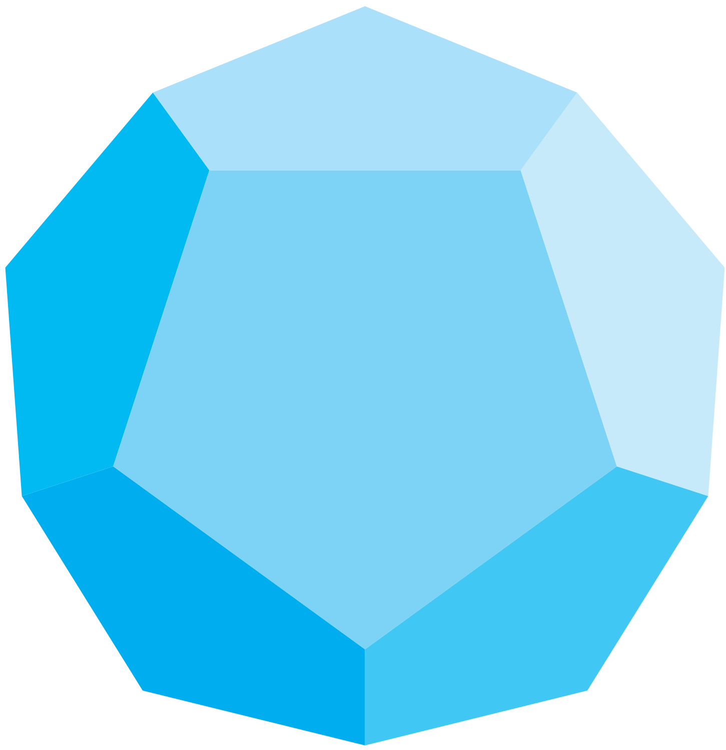

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Abstract

We introduce the notions of geometric and combinatorial reflection groups and discuss their geometric and combinatorial properties respectively. We introduce Coxeter groups contemporaneously with combinatorial reflection groups, and establish their equivalence. Via the construction of the so-called reflection representation of a Coxeter group, we shall show the equivalence of all three types of group, and so bring to bear the power of geometric intuition on the combinatorial questions of Coxeter groups. Following this we discuss two constructions of simplicial complexes from the combinatorial definition of a Coxeter group which link together the geometric and algebraic structure, and which are much easier to handle than the reflection representation. We shall establish some of their properties, and introduce the application to Tits buildings.

Preface

The study of Coxeter groups forms perhaps the most natural bridge between Algebra and Geometry. Subjects like Algebraic Geometry and Algebraic Topology have names which suggest that they fulfil this role; however their abstractness draws them away from the classical subject of Geometry, certainly as it was known to the ancients, since they deal with with the zero-loci of multi-variable polynomials and abstract topological spaces respectively. Instead Coxeter groups — which might more illustratively be called 'discrete groups generated by reflections' as H. S. M. Coxeter himself called them [12], or simply called abstract reflection groups — describe the symmetries of the geometric objects with which we are familiar, even from primary school in some cases: regular n-gons, tessellations or tilings of the plane, and platonic solids; but their study conveniently allows us to examine similar structures in arbitrary dimensions, and in non-flat geometries.

This gives some motivation as to why we might be interested in studying Coxeter groups from the point of view of Geometry. What of the Algebra? Certainly Algebra is indispensable in telling us about the behaviour of the Geometry, but is it merely a tool? The bridge between Geometry and Algebra goes both ways. Coxeter groups have some very interesting algebraic properties, particularly combinatorial. For example they admit (a number of) simple solutions to the Word Problem, and a solution to the Conjugacy Problem [22]. There are also a number of constructions of simplicial complexes purely from the combinatorial definition of the group, which then manifest the algebraic structure of the group geometrically, and which allow for simple geometric proofs of otherwise opaque and difficult combinatorial results.

The main aim of this report is to clearly highlight the interplay between Geometry and Algebra in the study of Coxeter groups. Indeed, the first two chapters are (almost) logically independent. In the first chapter we shall consider geometric reflection groups and how they act of a real vector space. In particular we shall see that they partition the space into regions called chambers, and the adjacency relations between these chambers forms a natural way to talk about the group structure. We shall develop the theory in the first instance for finite reflection groups, but later on generalise to infinite groups which act on the whole vector space. We shall see that almost all of the theory goes through, however there is a fundamental difference in the way we set up the reflections which causes some profound differences, and to highlight these, we discuss the two cases separately. In the second chapter we consider combinatorial groups, and derive a reasonable "axiom" which must be satisfied for a combinatorial group to be like a reflection group. We also give the definition of a Coxeter group as a combinatorial group, which at first seems largely unrelated to reflection groups. The main body of this chapter is devoted to studying some combinatorial properties of these Coxeter groups centred around a notion of length which we can define on the group. With some of this machinery under our belt, we are able to prove the equivalence of the definitions of a Coxeter group and a combinatorial reflection group.

The third chapter is the bridge between the first two chapters. Following J. Tits, we define a canonical "reflection" representation of a Coxeter group (on a real vector space) which looks very similar to the structures with which we were concerned in the first chapter. With some effort we shall be able to prove a simple looking theorem which has profound consequences, that the representation is an isomorphism between a Coxeter group and a geometric reflection group. This means that the theory we built up in the first chapter can be applied to Coxeter groups, in particular the intuition of Geometry can be brought to bear on abstract Algebra. The reflection representation will allow us to solve the Word Problem in Coxeter groups, and classify all finite Coxeter groups up to isomorphism. We finish this chapter by looking back at the combinatorial results we proved in the second chapter. With the language of chambers, and geometric intuition, we shall be able to reduce almost 10 pages of proofs to a single page.

Having seen the power of Geometry and the reflection representation, the fourth chapter is devoted to the construction of two related simplicial complexes which do the same job as the reflection representation, but are much easier to calculate and work with. In the first instance, the Coxeter complex corresponds to exactly the chambers of the reflection representation. The Coxeter group acts on it in a natural way by reflections, but crucially it can be defined in terms only of certain “special” subgroups of the Coxeter group. This lays bear to what extent the Coxeter complex encodes the algebraic structure of the group in geometrical structure. We shall be able to introduce an application of Coxeter complexes called buildings, which are very important in the study of Lie groups and algebras. They also relate closely to a very broad class of incidence geometries. The other simplicial complex is a later construction called the Davis complex. It is a refinement of the Coxeter complex by removing the messy infinite parts of a Coxeter group, and leaving only the well-behaved finite parts. This shares many of the characteristics of the Coxeter complex, but in particular admits a proper action of the Coxeter group by quasi Euclidean isometries, which the Coxeter complex does not.

This report comprises material drawn from many sources, and brings together some of the theory of Coxeter groups which is normally found confined to separate books. We have provided references throughout to results and proofs. Results where only the statement and not the proof is referenced carry my own proof; results with no reference are my own. We have provided notes at the end of each chapter mentioning specific instances. Most of the examples, and all of the diagrams in this report are my own, produced using the Tikz package in LaTeX. We have also provided an index and glossary of notation for the convenience of the reader. We have also not been shy in adding footnotes liberally throughout the text giving historical context, more detailed explanation, and further observations. With a few notable exceptions which are clearly signposted, the exposition throughout is my own. I have endeavoured to add justification and fill in the details of proofs which I thought we too brief, and have added many examples and diagrams throughout to help explain the ideas. The material is supported by extensive appendices on the background theory of simplicial complexes and circle inversions, which are particularly applicable to the final chapter . We have attempted to make the material accessible to an undergraduate with “reasonable” background, however there are a very few points where proofs require some more advanced Topology and Linear Algebra. These are kept to a minimum, and there will be little loss if these passages are skipped.

We shall finish this preliminary discussion by mentioning a few of the application and fields in which Coxeter groups arise. They are key to the study of Lie Groups and Lie Algebras, so much so that Bourbaki devotes chapters IV–VI of their discussion of these entirely to Coxeter groups [6], and indeed the classification of finite Coxeter groups and Simple Lie Algebras are very similar. Coxeter groups admit a partial ordering of their elements called the Bruhat order, which forms a natural language for the combinatorial properties of Coxeter groups, but also links closely to the geometry; the construction of certain cell complexes relating to the Coxeter group; and to Lie Groups [4, chapter 2]. The Bruhat order is also crucial to the definition of the so-called Kazhdan-Lusztig polynomials associated to a Coxeter group. These are a family of polynomials with integer coefficients indexed by pairs of group elements, and which play important roles in Algebraic Geometry and Representation Theory [4, chapter 5].

We have restricted ourselves in this last paragraph mainly to mentioning fields relating to the combinatorics of Coxeter groups, since the deeper theory we study in this text will lean towards the the more geometrical aspects.

Acknowledgements

My thanks go to my supervisor, Pavel Tumarkin, for his guidance throughout the project, and in particular for his valuable advice on the first draft of this report. Gratitude is also due to my fellow students on this project for there support, and for the collaboration in understanding ideas. In addition I would particularly like to thank Harsh Joshi for his constant companionships and support throughout this project; he consented to act as guinea pig for a lot of the way ideas are explained here, and he gave much good advice about the overall structure.

Contents

Notation

$\wedge$	The join of two simplicial complexes.
$\triangle(l, m, n)$	The triangle group with angles l, m, and n.
$\emptyset$	The empty set.
$\vert \cdot \vert$	The geometric realisation of a poset.
$\Vert \cdot \Vert$	The modulus of a vector or complex number.
$\sqcup$	Disjoint union.
$\lfloor \cdot \rfloor$	The floor function.
$\equiv_n$	Congruent modulo n.
$\#Y$	The cardinality of a set Y.
A_n	The Coxeter group of type A with n generators, isomorphic to S_{n+1}.
$\overline{A}$	The topological closure of A.
$\mathcal{A}$	A set of generators for a combinatorial group.
$\mathcal{A}^{-1}$	The set of inverses of the generators for a combinatorial group.
$\langle \mathcal{A} \mid \mathcal{R} \rangle$	A group presentation.
$\alpha^{\perp}$	For α a vector, the hyperplane orthogonal to α.
$\mathfrak{a}$	The set of apartments of a building.
$B(\cdot, \cdot)$	A symmetric bilinear form.
$B(z; r)$	The open ball in $\mathbb{C}$ centred at z of radius r.
$\mathbf{Bs}(\cdot)$	The barycentric subdivision of a simplicial complex.
C	A chamber.
$C \underset{H}{\mid} C'$	Two adjacent chambers separated by a wall H.
$\hat{\mathbb{C}}$	The complex numbers, together with the point at infinity.
Cöb	The set of comöbius transformations of $\hat{\mathbb{C}}$.
Con	The group of conformal transformations of $\hat{\mathbb{C}}$.
$\mathbf{Cone}(\cdot)$	The cone of a simplicial complex.
$\mathbb{C}^{\times}$	The set of non-zero complex numbers.
χ	An apartment in a building.
D_n	The dihedral group of order $2n$.
D_{∞}	The infinite dihedral group.
D_w	The set of minimal expressions for a group element w.
$d(C, D)$	The combinatorial distance between two chambers.
Δ	A simplicial complex.
$(\Delta, \mathfrak{a})$	A building.
ε	The empty word which acts as identity in a combinatorial group.
Fix_f	The fixed point set of a map f.
$Flag(\cdot)$	The set of flags of a poset.
f_i	The linear equation defining a hyperplane.
$Geom(\cdot)$	The geometric realisation of an abstract simplicial complex.
$GL(V)$	The general linear group on a vector space V.
Γ	A gallery.
γ	A circle in $\mathbb{C}$.
H	A hyperplane in $\mathbb{R}^n$, i.e. a codimension 1 linear subspace.
$H^{\pm}$	A half-space with respect to a hyperplane H.
$\mathcal{H}$	A collection of hyperplanes.

$\mathcal{H}'$ A collection of hyperplanes which define a chamber.

$\mathbb{H}^n$ n–dimensional hyperbolic space.

$\mathcal{I}$ The interior of the Tits cone.

I_γ Inversion with respect to a circle γ.

$\mathbb{I}$ The identity matrix.

$int(\cdot)$ The interior of a topological space.

Inv The set of circle inversions.

κ A colouring od a chamber complex.

K The fundamental chamber of the Davis complex.

L The nerve of a Coxeter system.

$l_S(\cdot)$ The length function on a Coxeter system.

M The Coxeter matrix of a Coxeter system.

Möb The group of Möbius transformations.

N The set of normals associated to a collection of hyperplanes.

$\mathcal{V}$ The Coxeter diagram of a Coxeter system.

$O(n, B)$ The orthogonal group on $\mathbb{R}^n$ with respect to a symmetric bilinear form B.

O_n The orthogonal group of Euclidean $\mathbb{R}^n$.

$\mathcal{P}(Y)$ The power set of a set Y.

$\varphi(\cdot)$ Euler's totient function.

$\mathbb{R}^+$ The strictly positive real numbers.

R The set of reflection in a Coxeter system.

$\mathcal{R}$ A set of relations for a combinatorial group.

$Re(\cdot)$ The real part of a complex number.

r A reflection in a hyperplane.

ρ The reflection representation of a Coxeter system.

ρ^* The dual of the reflection representation.

S A set of generators of a Coxeter group.

S^n The n-sphere.

S_n The symmetric group on n letters.

$\mathcal{S}$ The set of spherical subsets of S for a Coxeter system.

s A generator of a Coxeter group.

Σ The Davis complex of a Coxeter system.

σ^n An n–simplex.

T A subset of the generating set S of a Coxeter group.

t_i A letter in an expression for an element of a Coxeter system.

$\hat{t}_i$ A letter deleted from an expression for an element of a Coxeter system.

$(t_1, ..., t_d)$ A word in a combinatorial group.

U The Tits cone of the reflection representation.

V A finite dimensional real vector space.

V^* The dual of a vector space V.

$V^\perp$ The radical of a vector space V with a bilinear form.

W A group generated by reflections.

(W, S) A Coxeter system consisting of a Coxeter group W generated by a set S.

$W\mathcal{S}$ The set of spherical cosets of a Coxeter group W.

W_T The subgroup of a Coxeter group W generated by T.

X The Coxeter complex.

$\mathbb{Z}^+$ The strictly positive integers.

ζ_n A primitive n^{th}-root of unity.

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Chapter I

Geometric Reflection Groups

In 1935 H. S. M. Coxeter first classified all finite groups generated by reflections which act discretely [12]. We now call such groups “Coxeter groups” after him. The generalisation to studying infinite groups arising in the same way was largely done by J. Tits in the second half of last century. As motivated in the preface, while these groups can be studied purely algebraically and combinatorially, their association with Geometry is both historically and mathematically key to their study, and is also the source of much richness in the interpretation of results, and aesthetic beauty of the structures we can describe. For this reason we shall discuss geometric reflection groups in this chapter, postponing the formal definition of Coxeter groups until chapter II.

In this chapter we shall introduce formally what it means for a group to act discretely by reflections on a vector space, and discuss the differences between the finite and infinite case. We shall introduce the the language of chambers, adjacency, and galleries which will be vital later on. Moreover, with some very natural and mild conditions we shall be able explicitly show that the structure of these chambers is always simplicial.

I.1 Finite Reflection Groups

Everything we discuss will concern groups which act discretely, and so we shall normally omit to say this explicitly as we go through. To make up for this we shall start with a few words recalling exactly what this means, and how it will relate to our discussion. Consider the unit circle S^1 centred at 0 in $\mathbb{C}$, and the orientation preserving isometries which leave it unchanged. Now think of your favourite irrational number, maybe it is e; a rotation by $e\pi$ radians is just such an isometry fixing the circle. What is the orbit of the point 1? Since e is irrational, the orbit is infinite, because you will never return to the point 1, however there are points in the orbit as arbitrarily close to 1 as you like. This is typical of a non-discrete group action. Putting this another way, and more precisely, a group acting on a topological space is **discrete** if any point in any orbit of that action can be separated from the rest of its orbit by an open neighbourhood.

We are interested in groups generated by reflections. The rotation of the circle could have been achieved by the composition of reflections in two lines passing through 0 and intersecting at an angle of $\frac{1}{2}e\pi$ radians. This means that if we want our group to act discretely, then the reflections should be in mirrors which intersect at a dihedral angle which is a rational multiple of π. In the finite case we would need this anyway, but when we discuss infinite groups generated by reflections, we shall have infinite orbits, which we want nevertheless to be discrete, so this condition on intersecting mirrors will mean that we shall have to achieve infinite orbits using mirrors which are parallel.

In this section we follow the exposition of [8, chapter I]. We have rephrased his discussion, in general treating the geometrically "obvious" notions less rigorously, but where necessary, results are proved rigorously. We have attempted to make these proof more self-contained, and the argument clearer. We have included our own examples to help illustrate the theory.

1A Geometric Definitions

Definition I.1. Let V be a Euclidean vector space, which is to say $\mathbb{R}^n$ with the standard inner product. A **hyperplane** H in V is a co-dimension 1 linear subspace of V.

Remark I.1. Everything we deal with in this section will concern hyperplanes as linear subspaces. Why can we not include affine hyperplanes (hyperplanes which do not pass through the origin) as well to construct more finite reflection groups? The reason for this is that then our group generated by reflections would necessarily be infinite. If our group contained two distinct affine hyperplanes which did not intersect, they must be parallel, and reflection in one, followed by the other corresponds to a translation of V, which would be a group element of infinite order. If none of the affine hyperplanes are parallel, but they did not all pass through the same origin, consider three of them, then picture would be an n–dimensional version of figure I.1.

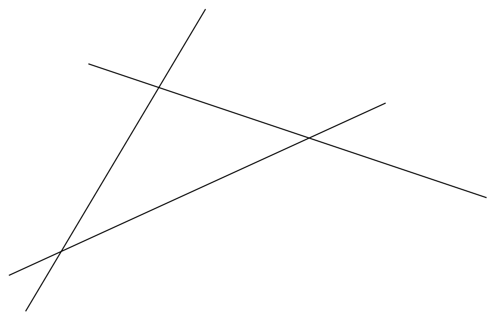


Figure I.1: A collection of three affine hyperplanes, intersected with a 2–dimensional plane.

The reflections just in those three lines would tile the plane with triangles, and so the corresponding group would necessarily be infinite. We shall generalise in this way in section I.2.

A hyperplane can, up to a sign, be uniquely identified with its unit normal vector at the origin α, so $H = \alpha^\perp$ (we shall remove the sign ambiguity in the proof of lemma I.1).

Definition I.2. The complement of H in V consists of two connected components, each called a **half-space**. The **reflection** of V with respect to H, typically denoted r_H, is the unique isometry of V which preserves H point-wise, and which swaps half-spaces. Explicitly this is given by

$$s_H(x) = x - 2(x \cdot \alpha)\alpha$$

where "$\cdot$" denotes the standard inner product.

If this is not familiar, try imagining it in two or three dimensions, and consider for example the image of α itself, noting that we chose α to be unit.

Definition I.3. Let W be a finite group of isometries of V generated by a finite set of hyperplanes $\{H_1, ..., H_k\}$, and let $\mathcal{H}$ be the smallest set of hyperplanes in V which contains this finite set, and which is stable under the action of W. Then W is a **finite reflection group**.

Note the fact that our definition of a hyperplane was as a linear *subspace*, which means that every hyperplane goes through the origin. How does this affect our discreteness condition? If $\mathcal{H}$ contains fewer than 2 hyperplanes, this question is vacuous, so assuming it contains at least two, all pairs of hyperplanes intersect at the origin at some dihedral angle, and from the preliminary discussion, we know that it must be a rational multiple of π.

We shall now give one more definition, which will help us to simplify our discussion. We noted above that we could, up to a choice of sign, identify a hyperplane H by its unit normal vector α. We have introduced a collection of hyperplanes $\mathcal{H}$, so equivalently, we could choose to specify a set of unit normal vectors

$$N := \{\alpha_H \in V \mid \|\alpha_H\| = 1,\ \alpha_H^{\perp} = H \in \mathcal{H}\},$$

where we assume that we have already chosen exactly one α corresponding to each H. If we set $V_0 := span_{\mathbb{R}} N$, then we can decompose V as $V_0 \oplus V_1$ for some complement V_1.

Definition I.4. We say that the group W generated by $\mathcal{H}$ is **essential** if V_1 is trivial, otherwise we say that W is **inessential**.

What does this mean geometrically? Heuristically, W is essential if V is as small as it can by, i.e. it has no invariant subspaces under the action of W. We can see that this is what being essential means, because for each H, α_H spans the eigenspace of r_H with eigenvalue -1, and every other eigenvalue is 1 (since r_H fixes H). This definition will simplify our discussion, because if we have a reflection group W which is inessential on V, without loss we can consider its restriction to V_0, on which it is essential.

Example I.1. Everything we have said so far has been in the generality of n-dimensions. We shall now give a 2–dimensional example which will hopefully explicate all of these ideas.

Consider $\mathbb{R}^2$ with $\alpha_1 = (1,0)$ and $\alpha_2 = (\cos(\pi/3), -\sin(\pi/3))$. Then these define two hyperplanes $H_1 = \alpha_1^{\perp}$ and $H_2 = \alpha_2^{\perp}$, which in $\mathbb{R}^2$ are lines through $(0,0)$ which intersect at an angle of $\pi/3$. We let W be the group generated by the two reflections in $H_1,\ H_2 \in \mathcal{H}$, call them s_1 and s_2 respectively. We know that $s_1 s_2$ and $s_2 s_1$ are rotations by $\pm 2\pi/3$, what about $s_1 s_2 s_1$? This is the conjugation of s_2 by s_1, so it is a reflection in a third line, H_3, which is the image $s_1(H_2)$. It is not hard to see that $\mathcal{H} = \{H_1, H_2, H_3\}$ is closed under the action of W, so this $\mathcal{H}$ is minimal as we required.

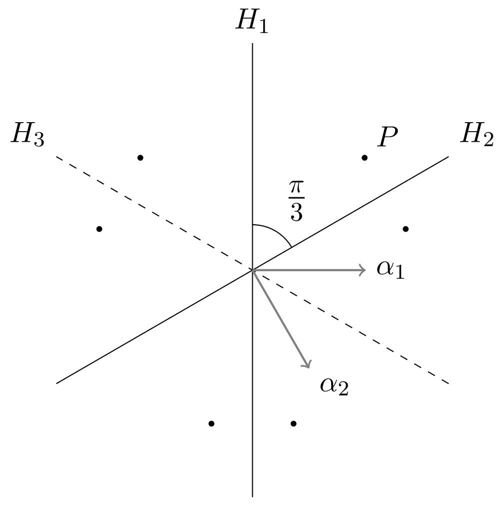


Figure I.2: A 2–dimensional discrete group generated by reflections in lines intersecting at angles of $\frac{\pi}{3}$. The orbit of a point P shown to illustrate discreteness.

Figure I.2 illustrates this example. We have marked on it a typical point P, and its orbit under W, which shows clearly that W is discrete. It is obvious from the diagram that W is essential: $s_1 s_2$ rotates the plane, so there are no non-trivial invariant subspaces under W.

What might an inessential version of W look like? Just extrude the diagram orthogonally out of the plane into $\mathbb{R}^3$. Then H_1 and H_2 would be defined by $\alpha_1 = (1,0,0)$ and $\alpha_2 = (\cos(\pi/3), -\sin(\pi/3), 0)$, and the subspace spanned by $(0,0,1)$ would be left invariant under the new W.

1B Examples

Let us work through some low dimensional examples of finite reflection groups.

1. If $V = \mathbb{R}^0 = \{0\}$, there are no hyperplanes, so $\mathcal{H} = \emptyset$, and $W \cong \{1\}$.
2. If $V = \mathbb{R}$, there is only one hyperplane, the point $\{0\}$, and reflection in this point just swaps the half-spaces $(-\infty, 0)$ and $(0, \infty)$, so $W \cong \{\pm 1\}$.
3. If $V = \mathbb{R}^2$, hyperplanes are the lines through the origin. If $\mathcal{H}$ contains just one line, then this line is an invariant subspace under W, so our reflection group would be inessential. As we want W to be essential, $\mathcal{H}$ must contain at least two lines. The picture will look like that in figure I.2. Choosing different angles for $\frac{\pi}{3}$, we get different groups: for each $n \in \mathbb{Z}$ choose $\frac{\pi}{n}$, then $\mathcal{H}$ will contain n lines. These are then the reflectional symmetry lines of the regular n-gon in the plane, centred at the origin. W is then the symmetry group of that n-gon, i.e. $W \cong D_n$, the dihedral group of order $2n$.
4. If $V = \mathbb{R}^3$, hyperplanes are just planes containing the origin. By an analogous argument to the one above, we deduce that $\mathcal{H}$ must contain at least 3 planes to be essential (although this is not a sufficient condition for 3–dimensions and up). Inspired by our observation about the case of $\mathbb{R}^2$, we can consider the regular polyhedra: the platonic solids. There are 5 of them, and each of their symmetry groups can be expressed as a finite reflection group in the way we have described.[1]

Remark I.2. We can convince ourselves that we have found all of the geometric reflection groups in $\mathbb{R}^0$ and $\mathbb{R}$, and perhaps guess that there is not anything more interesting in $\mathbb{R}^2$, but what about higher dimensions. There is no reason to assume that geometric reflection groups correspond exactly to the regular polytopes (indeed, they do not). We have been very restrictive in our assumptions at the start of section 1A, supposing that V is a *Euclidean* vector space. This is a nice assumption to make in low dimensions, since it allows us to visualise intuitively what is going on, but sooner or later we are going to have to relax this assumption if we want to consider all possible geometric reflection groups.

1C The Cell Structure of V with respect to $\mathcal{H}$

It should be obvious that $\mathcal{H}$ divides up V into polyhedral pieces (though they are unbounded). It will be useful to us later to have the language to discuss this structure in detail, and understand basic properties; this is what we shall cover for most of the rest of this chapter. This material is geometrically intuitive, so we shall state most results, taking their veracity to be more or less obvious. For a more rigorous development of these ideas, consult [8, section I.4].

Let V be a real inner product space of dimension n, and $\mathcal{H} = \{H_1, \dots, H_k\}$ an arbitrary finite collection of hyperplanes in V, with corresponding unit normals $\alpha_1, \dots, \alpha_k$. Each hyperplane H_i is then the vanishing set of the linear equation $f_i \colon V \mapsto \mathbb{R} \colon x \mapsto x \cdot \alpha_i$, i.e. $x \in H_i$ if and only if $f_i(x) = x \cdot \alpha_i = 0$ (where "$\cdot$" is the inner product). The two half-spaces with respect to H_i are then the points $x \in V$ such that either $f_i(x) > 0$ or $f_i(x) < 0$, write these as H_i^+ and H_i^- respectively.

[1] There is a very nice exposition on why there are only 5 platonic solids here: `youtu.be/2s4TqVAbfz4`. This also intuitively generalises the search for regular polytopes in arbitrary dimensions. One could use this in exactly the same way to construct essential geometric reflection groups in $\mathbb{R}^n$.

Definition I.5. A **cell** A in V is a non-empty subset of V obtained as the intersection of half-spaces and hyperplanes corresponding to elements of $\mathcal{H}$. In particular, choose a k-tuple of subsets of V, $(H_1^*, \ldots, H_k^*)$, where the i^{th} entry $H_i^* \in \{H_i, H_i^+, H_i^-\}$. Then $A = \bigcap_i H_i^*$ if this is non-empty.

Definition I.6. The **support**, L, of a cell A is its linear span (i.e. the span of all points in A), which is the minimal subspace of V containing A. We say that the **dimension** of a cell A is equal to the dimension of L, its support.

An equivalent way to characterise L is as the intersection of all the H_i's appearing in the definition of A (as opposed to $H_i^{\pm}$'s); if there are no H_i's in its definition, then we take $L = V$. A as a subset of L is defined by the strict inequalities $f_j(x) > 0$ or $f_j(x) < 0$ for $H_j^* = H_j^+$ or H_j^- respectively, hence A is open in its support.

Definition I.7. Writing $\overline{A}$ for the topological closure of A, we say that a cell B is a **facet** of A if $B \subseteq \overline{A}$ if and only if $\overline{B} \subseteq \overline{A}$. Following the practice of Bourbaki [6, p. 65], we reserve the name **face** for a facet B of A which is of dimension one less: $\dim B = \dim A - 1$. A cell whose support is V itself (or equivalently a cell of maximal dimension), is called a **chamber**.

Chambers partition $V \backslash \mathcal{H}$, and are in fact the connected components of this complement.

Definition I.8. We call the supports H_i of the faces of a chamber C, the **walls** of C.

H_i is a wall of C if and only if $\overline{C}$ and H_i intersect in a face of C. We have illustrated these ideas in figure I.3. The subset $\mathcal{H}' \subseteq \mathcal{H}$ of walls of a given chamber C, is the unique set of hyperplanes which define C.

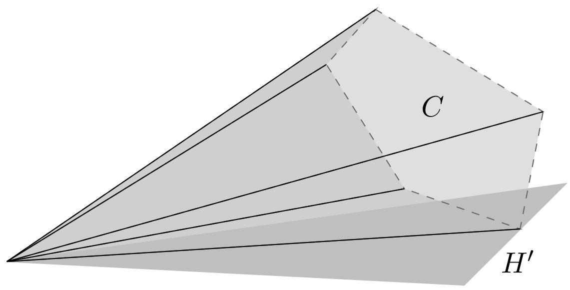


Figure I.3: A chamber in $\mathbb{R}^3$ which has 5 walls, the 5 faces of C can clearly be seen. The hyperplane H' is *not* a wall of C, even though the intersection with $\overline{C}$ has non-empty interior, since it is not the support of a face of C.

Let X be a partially ordered set (poset, see appendix B.1) of (open) cells defined by a collection of hyperplanes $\mathcal{H}$, and ordered by the facet relation (i.e. cells A and B satisfy $A \leq B$ if and only if A is a facet of B). Then any two cells A and B in X have a greatest lower bound, or meet, $int(\overline{A} \cap \overline{B})$ (that this is indeed another cell in the poset is not hard to see). We write this $A \wedge B$. Since all hyperplanes pass through the origin, the origin is a lower bound for any two cells in X.

Remark I.3. Every cell is a facet of a chamber (this is intuitively obvious, but is proved rigorously in [8, section I.4E, proposition 2]), and moreover, if it has co-dimension 1, it a face of exactly 2 chambers; indeed it is defined as the intersection of $n-1$ half-spaces and one hyperplane. Replacing this hyperplane by one of the two half-spaces it defines, then defines two chambers which it is a face of. That both of these intersections are non-empty follows from the fact that the original cell was non-empty.

Definition I.9. Two chambers are **adjacent** if the share a face, or equivalently, if their defining half-spaces are the same except for at most one $H_i^{\pm}$. Note that this means that every chamber is adjacent to itself. If two distinct chambers C and C' are adjacent, then there is a unique i such that H_i is a wall of both which separates them; it is the support of the common face, which can be expressed as $C \wedge C'$. We would then say that C and C' are **adjacent along** H_i.

A **gallery** is a sequence of chambers $\Gamma = (C_0, ..., C_d)$ in which consecutive chambers are adjacent. We say that Γ **connects** C_0 and C_d, writing $\Gamma : C_0, ..., C_d$. The **length** of Γ is d. If two consecutive chambers in a gallery are in fact the same chamber, we say Γ **stutters**.

Definition I.10. The **combinatorial distance** between two chambers C and D, $d(C, D)$ is the minimum over all galleries connecting C and D of the lengths of those galleries. A gallery which realises this length is called a **minimal gallery**.

This definition makes sense because there is always a gallery connection any two chambers. Such a gallery can be constructed inductively: if C and D are disjoint chambers, there is $H \in \mathcal{H}$ a wall of C which separates C from D, i.e. the two chambers are in different half-spaces with respect to H. Let C_1 be the chamber different from $C_0 := C$, which is adjacent to it along H. Proceeding likewise now with C_1 we shall construct the required gallery. As any gallery from C to D must necessarily cross all of the hyperplanes in $\mathcal{H}$ separating C and D, and the gallery constructed above crosses each of those exactly once, and does not stutter, we have in fact constructed a minimal gallery from C to D. From this follows the proposition below.

Proposition I.1.

1. *$d(C, D)$ is equal to the number of hyperplanes separating C and D,*
2. *if Γ is a minimal gallery connecting C and D, then it crosses each hyperplane separating C and D exactly once, and*
3. *if C and C' are distinct adjacent chambers, and D another chamber, then $d(C, D) = d(C', D) \pm 1$, with the sign determined by whether C' and D are on the same side of the wall which separates C and C' or not.* ■

[8, chapter 1, section 4E, proposition 4]

Definition I.11. The **diameter** of the poset of cells X, $diam(X)$ is the maximum over all pairs of chambers of $d(C, D)$.

The diameter of X is $k = \#\mathcal{H}$, which is realised by considering $d(C, -C)$, where $C = \bigcap H_i^+$, and $-C = \bigcap H_i^-$ is the chamber opposite to C. Then indeed C and $-C$ are separated by every hyperplane in $\mathcal{H}$.

In 1A we introduced the idea of a reflection group being essential. We can in fact give the same definition for the collection $\mathcal{H}$ without any mention of a group.

Definition I.12. We say that a non-empty collection of hyperplanes $\mathcal{H}$ is **essential** if $V_1 := \bigcap_{H_i \in \mathcal{H}} H_i = \{0\}$ (if we then consider a group W generated by $\mathcal{H}$, then W is essential if and only if $\mathcal{H}$ is essential, but for our new definition, we are not requiring the closedness property we did previously).

As before, we shall make the simplifying assumption that $\mathcal{H}$ is always essential. It is quite easy to convince yourself that necessarily $k = \#\mathcal{H} \geq n = \dim V$.

1D The Poset arising from W

We shall use the lexicon which we developed above to describe the action of a finite reflection group W on an inner product space V; a slightly more general set-up than we had in the first section. Recall that $\mathcal{H}$ is a finite collection of hyperplanes such that W is generated by the reflections s_H with $H \in \mathcal{H}$, and which is invariant under the action of W. That this is $W-$invariant is a consequence of the identity $ws_Hw^{-1} = s_{wH}$ for all $w \in W$. As above, let X denote the poset of cells coming from $\mathcal{H}$. From the definition of X one can see that reflections in any of the walls will permute the cells, and since reflection is a continuous map, the face relation will be preserved. Thus W acts on X by poset automorphisms (see definition B.3). Now follows a very important theorem about the action of W on X.

Theorem I.1.

1. *W acts simply-transitively*[2] *on the chambers of X, hence the number of chambers is equal to $\#W$.*
2. *W is generated by the reflections in the walls of any fixed chamber C in X.*
3. *The collection of hyperplanes $\mathcal{H}$ associated with W necessarily consists of all hyperplanes H such that reflection in H, s_H, is an element of W.*

Remark I.4. *A priori* it seemed that X depended on the choice of $\mathcal{H}$, however the last claim of this theorem shows that X is dependent only W and the associated vector space V.

Proof. The proof proceeds in six steps.
Step 1:

Let C be a chamber with wall H, and let t be reflection with respect to H. Then tC and C are adjacent along H, and moreover they are distinct.

Indeed, let A be the face of C supported by H, then the corresponding face of tC is $tA \subseteq sH$, but by definition of t, it fixes H, so $tH = H$, and $tA = A$, thus C and tC share a face supported by H. The last claim is obvious.
Step 2: Fix a chamber C and let S be the set of reflections t_H such that H is a wall of C.

For all $w \in W$ and $t_H \in S$, wt_HC and wC are distinct and adjacent along wH.

Indeed, this follows from step 1 by applying the action of w. We can illustrate this using the following diagram:

$$C \underset{H}{|} tC \quad \overset{w}{\longmapsto} \quad wC \underset{wH}{|} wtC$$

Step 3:

Given $t_1, \ldots, t_d \in S$, the gallery

$$\Gamma : C, t_1C, t_1t_2C, ..., t_1t_2 \cdots t_dC$$

is non-stuttering, and any non-stuttering gallery from C has this form.

Indeed, the first claim follows from step 2. Conversely let $\Gamma : C_0, ..., C_d$ be a non-stuttering gallery, with $C_0 = C$. We proceed by induction on d. There is nothing to prove if $d = 0$, so let $d \geq 1$ and assume that the gallery $\Gamma' : C_0, ..., C_{d-1}$ has the required form. C_{d-1} and C_d are distinct and adjacent along some wall H of $C_{d-1} = t_1 \cdots t_{d-1}C$. Let H' be the corresponding wall of C, that is choose H' such that $t_1 \cdots t_{d-1}H' = H$, and let t_d be reflection with respect to H'. Then by step 2

$$C \underset{H'}{|} t_dC \quad \overset{t_1 \cdots t_{d-1}}{\longmapsto} \quad C_{d-1} \underset{H}{|} t_1 \cdots t_{d-1}t_dC$$

[2] A group G acts simply-transitively on a set Y if for all $y, y' \in Y$ there exists a unique $g \in G$ which takes y to y'.

and since $C_{d-1} \underset{H}{|} C_d$ by definition of H, and we know that C_{d-1} is distinct from $t_1 \cdots t_{d-1} t_d C$ and C_d. In view of remark I.3 we conclude that $t_1 \cdots t_{d-1} t_d C = C_d$ as required.

Step 4: Let W' be the subgroup of W generated by S. By step 3 we know that W' acts transitively on the chambers of X, since any chamber can be connected to a chosen starting chamber C by a gallery; and so has the form $t_1 \cdots t_d C$. Hence $W \supset W'$ acts transitively on the chambers.

$W = W'$; that is claim *2* of the theorem holds.

Indeed, it suffices to show that W' contains all the reflections t_H in W where $H \in \mathcal{H}$. Every hyperplane $H \in \mathcal{H}$ is the wall of some chamber, call it D. We know that $D = wC$ for some $w \in W'$, so $H = wH'$ where H' is a wall of C. Then we can write $t_H = w t_{H'} w^{-1} \in W'$ as required.

Step 5: We have shown that W acts transitively, to see that it acts simply-transitively, and so prove *1*, we must show that the stabiliser of C in W is trivial.

Let $w \in W$ be a non-trivial element, then $wC \neq C$.

Indeed, let $w = t_1 \cdots t_d$ be an expression for w with minimal "length" d, and consider the gallery $\Gamma : C, w_1 C, ..., w_d C$, where $w_i = t_1 \cdots t_i$. We shall show that if s_1 is reflection with respect to the wall H_1 of C, then H_1 separates C and wC. In general let H_i be the wall of C fixed by t_i, and write $C_i = w_i C$. By step 2 we have

$$C \underset{H_i}{|} t_i C \quad \overset{w_{i-1}}{\longmapsto} \quad w_{i-1} C \underset{w_{i-1} H_1}{|} C_i$$

Note that the wall separating two consecutive chambers in Γ (which by step 3 we know is non-stuttering), is the unique hyperplane in $\mathcal{H}$ separating these chambers, by the argument in remark I.3. It follows that H_1 must separate C from wC unless Γ crossed H_1 more than once[3]. Assume therefore that C and wC are on the same side of H_1, so $w_{i-1} H_i = H_1$ for some $i > 1$. Considering the associated reflections, we get

$$\begin{aligned} w_{i-1} t_i w_{i-1}^{-1} = t_1 &\implies w_{i-1} t_i = t_1 w_{i-1} \\ &\implies t_1 \cdots t_{i-1} t_i = t_1 t_1 t_2 \cdots t_{i-1} = t_2 \cdots t_{i-1} \end{aligned}$$

and hence $w = t_1 t_2 \cdots t_d = \hat{t}_1 t_2 \cdots \hat{t}_i \cdots t_d$, where $\hat{t}_i$ means that t_i has been deleted from the expression for w, but this contradicts the minimality assumption, whence the claim.

Step 6: The last step is to prove *3*.

If H is a hyperplane fixed by a reflection $t \in W$, the $H \in \mathcal{H}$.

Indeed, assume that $H \notin \mathcal{H}$, so that $H \not\subset \bigcup_{H' \in \mathcal{H}} H'$ because a linear subspace cannot be a finite union of proper subspaces $H \cap H'$. Hence H meets some chamber, call it D. Since t fixes H, and tD meets D, we must have $H = tH$, which contradicts step 5. [8, chapter I, section 5A] ■

Remark I.5. The number of chambers in X is equal to the number of elements of W, and W acts transitively on these chambers. A simple geometric argument shows that every chamber must have the same number of walls. If we fix a chamber C, and label this with the identity element in W, ε, then we can label every other chamber of X by the unique element of W which maps this fundamental chamber to that chamber. With S as in step 2 of the proof, we know that S generates W, and then galleries from C to the chamber wC, labelled w, correspond

[3]In particular, an even number of times; however we do not need this.

to expressions for w in the generators S, and so minimal galleries correspond to "minimal expressions", as mentioned in step 5 [8, remark 2, p. 18].

In the proof of step 5, there was nothing special about H_1, the same argument proves in fact that if Γ is a gallery from C to wC, then every wall crossed by Γ an odd number of times separates C from wC. This way of talking about group elements, and this method of reasoning will be core in the latter part of chapter III.

Example I.2. Consider the dihedral group D_4 acting on Euclidean $\mathbb{R}^2$ in the usual way, analogous to example I.1. The cell structure can be seen in figure I.4. There are 8 chambers just as there are 8 group elements, as *(1)* in the theorem states. We choose one of these chambers to be the fundamental chamber C, and label its walls H and H'. If s and s' respectively are the reflections in these walls, then $S = \{s, s'\}$ generates D_4. As described in the remark above we can label each of the chambers by the group element which maps C to it.

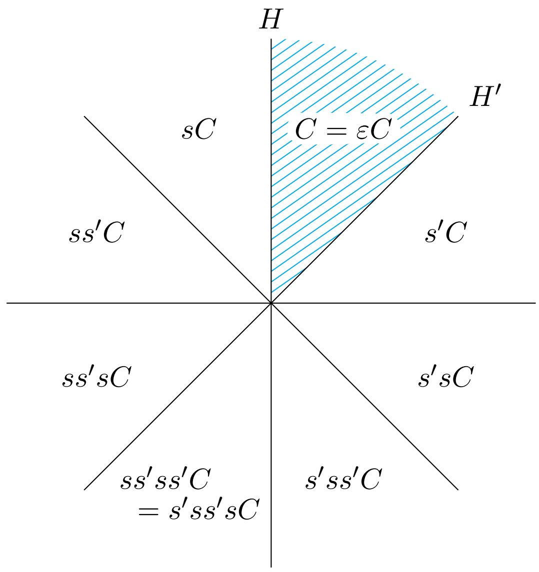


Figure I.4: The labelled chambers in the poset X corresponding to the group D_4.

1E Chambers

Definition I.13. Let G be a group acting on a set Y. A connected subset F of Y is called a **fundamental domain** for the action of G if F contains a representative point from every orbit space in Y, and that representative is unique except possibly on the boundary of F.

Example I.3. Consider the real numbers acting on the unit disc in $\mathbb{C}$ via $z \mapsto e^{ix}z$ for all $x \in \mathbb{R}$. The reals act as rotations, and the orbit spaces are the concentric circles around 0. An example of a fundamental domain for this action is the line segment between 0 and 1 in the complex plane, as this intersects each concentric circle exactly once.

Fundamental domains allow us to visualise the orbit space. Since they are connected, it is easy to think of the fundamental domain "tiling" the space under the action of the group. From this description, and statement *(1)* of the theorem, it is easy to show that a fundamental domain for the action of W on X is $\overline{C}$, for C any chamber [8, chapter I, section 5F, theorem], in particular we might as well take C to be the fundamental chamber. We need to include the boundary of C so that we capture every orbit space, see definition I.7. In the example above one can clearly see how $\overline{C}$ is a fundamental domain for the action of D_4, and how this action "tiles" $\mathbb{R}^2$ with copies of C. From the preceding it should be evident that chambers can tell us a lot about the behaviour and structure of a finite geometric reflection group[4]. Can we say something about the chambers associated to a finite reflection group in general? The remainder of this section will be devoted to answering this question.

From the definition of cells, it is clear that they partition V into disjoint convex connected sets which are closed under scalar multiplication by positive real numbers. Geometrically this means that they are a cone over some $(n-1)$–dimensional polytope. So in figure I.3, C is the cone over a pentagon, each of its faces are a cone over an open line segment, and the one dimensional facets (what we might call edges), are cones over a point. We know that the dimension of a chamber C is $n = \dim V$.

Definition I.14. We say that C is a **simplicial cone** if it is a cone over an $(n-1)$–simplex. (Recall: an $(n-1)$–simplex σ^n in $\mathbb{R}^n$ has n vertices $\{e_1, \ldots, e_n\}$ which are in general position, then $\sigma^n = \{\sum_i \lambda_i e_i \mid \lambda_i > 0,\ \sum_i \lambda_i = 1\}$.)

If C is simplicial over an $(n-1)$–simplex with vertices $\{e_1, \ldots, e_n\}$, then we can write $C = \{\sum_i \lambda_i e_i \mid \lambda_i > 0\}$ by the definition of a cone. Our first lemma characterises when a chamber C is a simplicial cone in terms of the number of its walls.

Lemma I.1. *Let C be a chamber in V with respect to an essential collection of hyperplanes $\mathcal{H}$, then C is a simplicial cone if and only if C has exactly n walls.*

Proof. Suppose C is a simplicial cone, then its faces are

$$\left\{ \sum_{i=1}^{n} \lambda_i e_i \middle| \lambda_i > 0 \text{ if } i \neq j,\ \lambda_j = 0 \right\}$$

for $j = 1, \ldots, n$; so C indeed has exactly n walls.

Now suppose C has n walls, i.e. $\mathcal{H}' = \{H_1, \ldots, H_n\}$. Since $\mathcal{H} = \{H_1, \ldots, H_k\}$ is essential, $V_1 = \bigcap_{H_i \in \mathcal{H}} H_i = \{0\}$. V_1 is a facet of every cell, so in particular it is a facet of C; hence $V_1 = \bigcap_{H_i \in \mathcal{H}'} H_i = \{0\}$, and $\mathcal{H}'$ is also essential. Recalling that we called the defining equations of the H_i's f_i, this observation implies that the system of simultaneous equations $f_1(x) = \cdots = f_n(x) = 0$ has only a trivial solution, hence $\{f_i\}_{i=1}^n$ is a basis of the dual space V^* of V. Let $\{e_i\}_{i=1}^n$ be the basis of V dual to $\{f_i\}_{i=1}^n$, i.e. $f_i(x) =: e_i \cdot x$. Suppose we choose the f_i's so that the points of C are given as the simultaneous solutions of $f_1(x), \ldots, f_n(x) > 0$ (which choice we have the freedom to make, it is the same choice as when we labelled the half-planes with respect to H_i as H_i^+ or H_i^-). Now we have $x \in C$ if and only if $f_i(x) > 0$ for $1 \leq i \leq n$ if and only if $x = \sum_i \lambda_i e_i$ for $\lambda_i > 0,\ 1 \leq i \leq n$, that is C is a simplicial cone [8, chapter I, section 4C, proposition]. ■

Notice how in this proof we simplified our notation by assuming that C was given by the positive inequalities $f_i(x) > 0$. From now on, if we have a particular chamber C under discussion, we shall assume that we have chosen the f_i's in this way. We shall call such a chamber the **fundamental chamber**. We also chose to write $f_i(x) = e_i \cdot x$. The vector e_i (which we were formerly calling α_i), is a normal to H_i, so our method of fixing the f_i's is just the same as choosing e_i such that it points "towards" C. Finally note that taking a non-zero multiple of f_i does not change the definition of H_i, so we might as well choose the e_i's to be unit vectors.

[4]In fact they tell us everything, because from the chambers X can be reconstructed, and from X the whole of W, since W is generated by the reflections in the walls in X if any one of the chambers, see [8, chapter I, appendix D] for details.

We noted in section 1B that our opening assumptions were quite restrictive on the sorts of reflection groups we might be able to produce. We shall begin to tackle these shortcomings now. Instead of assuming V is $\mathbb{R}^n$ with the standard Euclidean inner product, let us now assume that it is equipped with an inner product denoted $B(\cdot,\cdot)$, and that $f_i(x) = B(e_i, x)$. We are ready to state our second lemma, which gives us a sufficient condition for C to be a simplicial cone.

Lemma I.2. *Let $\mathcal{H}$ be an essential collection of hyperplanes in V, and let C be a chamber with respect to $\mathcal{H}$, with walls $\mathcal{H}' = \{H_1, \ldots, H_r\}$. If $B(e_i, e_j) \leq 0$ for all $1 \leq i \neq j \leq r$, then C is a simplicial cone.*

Proof. Since $\mathcal{H}$ is essential, we know that $r \geq n$, and $\{e_i\}_{i=1}^r$ spans V; by the previous lemma, we know that for C to be a simplicial cone, we need $r = n$, so it is sufficient to show that $\{e_i\}_{i=1}^r$ is linearly independent.

Suppose to the contrary that there exists a non-trivial linear combination $\sum_{i=1}^r \lambda_i e_i = 0$, and choose such a linear combination which has a minimal number if λ_i's non-zero[5]. Then we claim that all non-zero λ_i's must have the same sign. Suppose some are negative and some positive, we can split up the sum as $\sum_{i \in I} \mu_i e_i = \sum_{j \in J} \mu_j e_j$, where I and J are disjoint non-empty subsets of $\{1, \ldots, r\}$, and all μ's are strictly positive. Then

$$0 \leq B\left(\sum_{i \in I} \mu_i e_i, \sum_{i \in I} \mu_i e_i\right) = B\left(\sum_{i \in I} \mu_i e_i, \sum_{j \in J} \mu_j e_j\right) = \sum_{i \in I}\sum_{j \in J} \mu_i \mu_j B(e_i, e_j) \leq 0$$

where the first inequality comes from positive-definiteness B, and the second from the hypothesis of the lemma. We conclude that $\sum_{i \in I} \mu_i e_i = \sum_{j \in J} \mu_j e_j = 0$, and since we chose our original linear combination to be minimal, all μ's and hence all λ_i's are zero, a contradiction. Hence a minimal non-trivial linear combination $\sum_{i=1}^r \lambda_i e_i = 0$ will have all λ_i's with the same sign. We can take them all to be positive.

Finally we shall use that the inequalities $B(e_i, \cdot) > 0$ for $1 \leq i \leq r$ define C which is non-empty, i.e. $\emptyset \neq C := \{x \in V \mid B(e_i, x) > 0 \ \forall i\}$, so there exists $x \in C$, which we can take the inner product with the linear combination to get

$$0 < \sum_{i=1}^r \lambda_i B(e_i, x) = B\left(\sum_{i=1}^r \lambda_i e_i, x\right) = B(0, x) = 0$$

a contradiction, hence our assumption that there exists a non-trivial linear combination of e_i's which vanishes was incorrect, and so they are linearly independent, as required. [8, chapter I, section 4D, proposition] ■

The condition that $B(e_i, e_j) \leq 0$ says that the angle between e_i and e_j must not be acute with respect to B. Figure I.3 shows a 3–dimensional chamber which has 5 walls and so is not a simplicial cone. It is clear that the angles between walls are obtuse (with respect to the standard inner product), and so the angles between the corresponding e_i's are acute. Neither of the previous two lemmata require $\mathcal{H}$ to arise from a finite reflection group, they hold for any finite collection of hyperplanes.

Theorem I.2. *Let W be a finite reflection group acting essentially on a vector space V, and let X be the associated poset of cells in V. Then every chamber in X is an open simplicial cone.*

[5] This assumption is required to get the first contradiction below. K. Brown fails to do this in his proof [8] so it is not quite rigorous. Our thanks to P. Tumarkin for this observation.

Proof. Let C be a chamber in X with walls $\{H_1, ..., H_r\}$, each of which has corresponding unit normal e_i, chosen to point "towards" C, that is $C = \bigcap_i H_i^+$. Let s_i be reflection in H_i, so by *2* of theorem I.1, $S = \{s_1, ..., s_r\}$ generates W. Write m_{ij} for the order of $s_i s_j$ in W. We shall prove that

$$B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right) \tag{I.1}$$

So that for $i \neq j$, $B(e_i, e_j) \leq 0$, and hence by the previous lemma, C (which by definition is open), is a simplicial cone.

The equation clearly holds if $i = j$ since each s_i is a reflection, and hence has order 2, so fix i and j distinct. Let $W' = \langle s_i, s_j \rangle$ be the subgroup of W generated by s_i and s_j which acts essentially on the 2–dimensional subspace of V, $V' = \mathbb{R}e_i \oplus \mathbb{R}e_j$. Since the order of $s_i s_j$ is unchanged when we consider the restriction to V', the equation above will hold if and only if it holds for B restricted to V' as well.

Let $\mathcal{H}' = \{w'H \cap V' \mid w' \in W', H = H_i \text{ or } H_j\}$ be the minimal collection of lines in $\mathcal{H} \cap V'$ containing $H_i' = H_i \cap V'$ and $H_j' = H_j \cap V'$ which is invariant under W'. $\mathcal{H}'$ is the canonical set of hyperplanes associated to the action of W' on V'. Writing $C' = C \cap V'$, the walls of C' are the lines H_i' and H_j', which have unit normals e_i and e_j which still point "towards" C'.

Let $m = \#\mathcal{H}'$ which is at least 2. The $\mathcal{H}'$ divides the plane V' into $2m$ chambers, each of which is a sector bounded by two rays. Since W' acts transitively on these chambers ((*1*) of theorem I.1), and reflections preserve angles, all of the sectors are congruent, in particular each has angle $\frac{2\pi}{2m} = \frac{\pi}{m}$. W' is generated by the reflections in the walls of C', H_i' and H_j', which intersect at that angle; hence W' is the dihedral group of order $2m$, see example 3 in section 1B, and the order of $s_i s_j$ is m. We then have that the angle between e_i and e_j is $\pi - \frac{\pi}{m}$, and so the angle formula for inner products gives

$$B(e_i, e_j) = \|e_i\|\|e_j\| \cos\left(\pi - \frac{\pi}{m}\right) = -\cos\left(\frac{\pi}{m}\right)$$

where $m = m_{ij}$ as required. Note that the "$-$" sign appears because of our convention about the direction we chose for each e_i. [8, chapter I, section 5C, theorem] ■

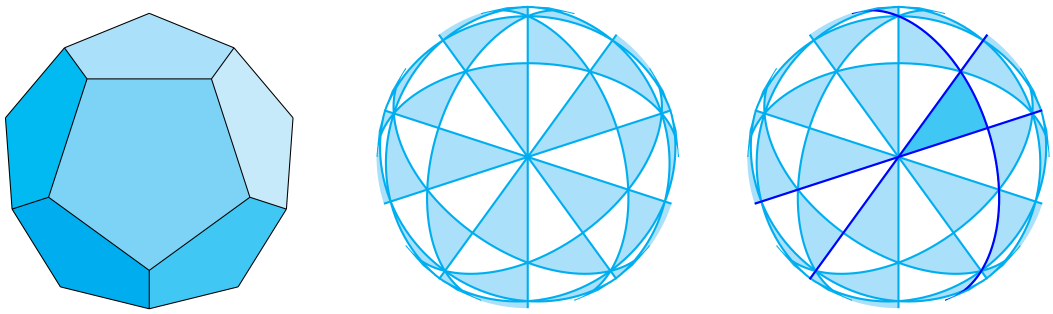
We shall return to the poset X in chapter IV, where we shall study it more formally in terms of its algebraic description instead the geometric one we have defined here. We prove in fact that it is a simplicial complex (we have so far been careful only to call it a poset), which should not be too much of a surprise in light of theorem I.2: that the chambers of X are simplicial cones. We shall however make one observation which we shall come back to at the end of chapter III. We noted in remark I.1 that a necessary condition for a geometric reflection group to be finite was that all the hyperplanes in $\mathcal{H}$ pass through the same point, taken to be the origin. This, together with the result that all of the chambers of X are simplicial cones, means that if we intersect X with the unit $(n-1)$–sphere in V (were V has dimension n), we shall get a triangulation of the sphere: all of the 1–dimensional facets become 0–simplicies, all of the faces become $(n-2)$–simplicies, and all of the chambers become $(n-1)$–simplicies. If we do this in the case of the example of the above, intersecting the cells with the unit circle centred at the origin, we get a triangulation of S^1 by the regular octagon. We give a 3–dimensional example below.

Example I.4. In section 1B we suggested that one way to construct finite reflection groups would be to consider the symmetry groups of regular polytopes. We shall now do this explicitly for the dodecahedron.

From the discussion above we know that we might as well project onto the sphere. If we mark on all of the reflection planes of the dodecahedron as great circles we get figure I.5b. This is a periodic tiling of the sphere by triangles which have angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{5}$. Even without theorem I.1 we can argue that the symmetry group of the dodecahedron acts transitively on these triangles (which are the simplicies over which the chambers are simplicial cones), as seen in the previous theorem. Indeed rotations of the dodecahedron will take any pentagonal face to any other, and the symmetries of the pentagon mean that within a pentagon, any triangle can be taken to any other.

Choosing a fundamental chamber, the symmetry group is generated by the reflections in the walls of this chamber, see figure I.5c. Reflection in one wall followed by another gives a rotation, the order of which is given by the denominator of the dihedral angle between them, and so if we call these reflections a, b, and c, the symmetry group is given by

$$\langle\, a, b, c \;\mid\; a^2 = b^2 = c^2 = (ab)^5 = (ac)^3 = (bc)^2 = \varepsilon \,\rangle$$



(a) (b) (c)

Figure I.5: Calculating the symmetry group of the dodecahedron (a). The planes of symmetry intersected with the unit circle are shown in (b). After making the choice of a fundamental chamber, the symmetry group is generated by the reflections in its walls (c).

I.2 Infinite Reflection Groups

In the previous section, we made a relatively detailed study of the way in which finite reflection groups act on real vector spaces, and in particular we saw the importance of chambers in understanding such actions. We shall now turn to consider infinite reflection groups. We shall see that almost all of the definitions and results either carry straight over to the infinite case, or have a direct analogue.

In remark I.1 we justified why we considered only hyperplanes which were *linear* subspaces of the vector space, because otherwise we would necessarily get an infinite group. The converse to this is that we shall necessarily have affine hyperplanes if we want to have an infinite geometric reflection group *which acts discretely*, since the collection of hyperplanes associated to such a group is infinite, and if they all passed through the same point in a finite dimensional vector space, then the group action would not be discrete. Formally we would say that for the group action to be discrete, the collection of affine hyperplanes must be **locally finite** [5, chapter 1, section 1, definition 8], which means that only finitely many affine hyperplanes intersect at any given point.

Definition I.15. Let V be a finite dimensional real vector space, which we shall assume throughout the following has an inner product B. An **affine hyperplane** in V is a translate of a hyperplane of V. The **affine space** of V, denoted $\mathrm{Aff}(V)$ is V along with all affine hyperplanes in V (this takes the vector space V and then “forgets” the origin).

Every affine hyperplane H in V divides V into two open connected components, called **half-spaces**; we call the unique isometry of V which fixes H point-wise and swaps half-spaces the **reflection** of V in H.

Definition I.16. Let W be an infinite group of isometries of V generated by a finite set of affine hyperplanes $\{H_1, ..., H_k\}$, and let $\mathcal{H}$ be the smallest set of hyperplanes in $\mathrm{Aff}(V)$ which contains this finite set, and which is stable under the action of W. If $\mathcal{H}$ is locally finite, then W is an **infinite reflection group**.

In the previous section, we gave two definitions of "essential", one for a group, and one for a collection of hyperplanes. Since we are now working in an affine space, the second definition no longer works, however we can trivially modify the first to work in this setting.

Definition I.17. Suppose the group W is generated by reflections in the affine hyperplanes $\{H_1, ..., H_k\}$, and that H_i has a unit normal e_i. W is **essential** if $\{e_1, ..., e_k\}$ span V; otherwise we say that W is **inessential**.

We illustrate the ideas with the following example.

Example I.5. What is the simplest example of an infinite reflection group? There are no affine hyperplanes in $\mathbb{R}^0$, so let $V = \mathbb{R}$. The affine hyperplanes in V are its points. From example 2 in section 1B we know that one affine hyperplane generates a finite reflection group, so let W be generated by reflection in two points: 0 and 1. Reflection in one point followed by reflection in the other preserves the ordering on $\mathbb{R}$, and takes $0 \mapsto 2$. Since it is an isometry, this must be a translation by 2 in the positive direction. It follows that the minimal set $\mathcal{H}$ must be the set of integers. W then acts by translations of $\mathbb{R}$ by integer multiples of 2, and by reflections in the integers.

If one added a point at infinity, the extended real line is homeomorphic to a circle, and translations behave "like" rotations of the circle (save that ∞ is fixed). since a circle is like a regular ∞-gon, this group is often called the infinite dihedral group, and denoted D_∞. The connection between D_n and D_∞ will be clearer after example II.4.

As with example I.1, we can imagine an inessential version of this example. Embed $\mathbb{R}$ into $\mathbb{R}^2$, then the affine hyperplanes are the lines orthogonal to $\mathbb{R}$, intersecting at each integer. The unit normals of these hyperplanes all lie in $\mathbb{R}$, and hence do not span $\mathbb{R}^2$ meaning that The group is now no longer essential.

All of our definitions regarding the cell structure of V with respect to $\mathcal{H}$: walls, chambers, the poset X *et cetera*, carry through to the affine case by replacing "hyperplane" with "affine hyperplane", and similar small alterations. Even the definition of "diameter" if we say that it is infinite if no maximum combinatorial distance exists (although trivially X will always infinite diameter if it corresponds to an infinite reflection group).

Even though there will be an infinite number of chambers in X if W is an infinite reflection group, any two chambers will be connected by a finite gallery, since our discreteness condition means that any two chambers must be separated by only a finite number of walls. Then the same arguments as before mean that proposition I.1 still holds in this more general setting. Examining the proof of theorem I.1, we see that we need only change "linear subspace" to "affine subspace" in step 6 to get the same result in the infinite case, provided that we interpret $\#W$ in the theorem statement as the cardinality of W, which is always countably infinite, since W is finitely generated by assumption. This also means that the comments in remark I.5 also hold in the infinite case, so we can label the chambers of X by a unique element of W each after the choice of a fundamental chamber.

In section 1E we proved that the chambers of a finite reflection group are simplicial cones. What is the analogous result for infinite reflection groups? First we need a definition which will be very useful in our later discussion, and which will be explored more fully in section 2C of chapter II; for now we shall just state it without any justification for why it is a sensible definition to make.

Definition I.18. Let W be a geometric reflection group (finite or infinite), generated by (affine) hyperplanes $\{H_1, ..., H_k\}$. If there is a partition of $\{1, ..., k\}$ into non-empty sets I and J such that for any $i \in I$ and $j \in J$, H_i and H_j intersect orthogonally, then W is **reducible**; if no such partition exists W is **irreducible**.

Theorem I.3. *Let W be an essential and irreducible infinite reflection group acting on a vector space V, and let X be the associated poset of cells in V. Then every chamber in X is an open simplex.*

For the proof we shall first need a lemma from linear algebra which we state without proof.

Lemma I.3. *Suppose $\{e_1, ..., e_n\}$ span V and are linearly <u>dependent</u>; if*

1. *$B(e_i, e_j) \leq 0$ for all $i \neq j$, and*
2. *there is no partition of $\{1, ..., n\}$ into non-empty sets I and J such that for any $i \in I$ and $j \in J$ $B(e_i, e_j) = 0$,*

then $n = \dim V + 1$ and there exist $c_i > 0$ such that $\sum_i c_i e_i = 0$, and for any linear combination $\sum_i c'_i e_i = 0$, there exists ξ such that $c'_i = \xi c_i$. [6, chapter V, section 3, lemma 5]

Corollary I.1. *Let W be an essential and irreducible infinite reflection group acting on a vector space V and generated by affine hyperplanes $\{H_1, ..., H_k\}$. Let X be the associated poset of cells in V. Then each chamber C has $\dim V + 1$ walls, and if e_i is the unit normal vector to H_i pointing towards C for all i, there exist $c_i > 0$ such that $\sum_i c_i e_i = 0$, and for any linear combination $\sum_i c'_i e_i = 0$, there exists ξ such that $c'_i = \xi c_i$.*

Proof. Fix a chamber C. With notation as in definition I.17, since W is essential, $\{e_1, ..., e_k\}$ spans V. Suppose e_i and e_j correspond to distinct walls of C. If those walls are parallel, $e_i = -e_j$, and so $B(e_i, e_j) = -1$. If the corresponding walls intersect, then the proof of theorem I.2 shows that $B(e_i, e_j) \leq 0$. Hence the first condition of lemma I.3 is satisfied. Because W is irreducible, the second condition is also satisfied.

If $\{e_1, ..., e_k\}$ were linearly independent then they would form a basis of V, so after an affine transformation of V, $\{H_1, ..., H_k\}$ could be identified with a set of mutually orthogonal hyperplanes through the origin ("coordinate hyperplanes") so they would all share a common point, the origin. Hence the original affine hyperplanes $\{H_1, ..., H_k\}$ must share a common point which is invariant under W. But then by remark I.1 and the discreteness assumption, W would be finite, a contradiction. Hence $\{e_1, ..., e_k\}$ are linearly dependent so we can apply lemma I.3. (Adapted from [6, chapter V, section 3, proposition 8]) ■

Proof of theorem I.3. Fix a chamber C, and let $\{H_0, ..., H_d\}$ be the walls of C. Write e_i for the unit normal vector to H_i pointing towards C for each i. By corollary I.1, $\{e_1, ..., e_d\}$ (excluding e_0) form a basis of V, and so by the argument in the second paragraph of the proof of that result, there is a point $a \in \bigcap_{i=1}^{d} H_i$. Let us re-define V so that it has origin a, and $\{H_1, ..., H_d\}$ are *linear* hyperplanes, and H_0 is an affine hyperplane. There is a basis $\{e'_1, ..., e'_d\}$ of V which satisfies $B(e_m, e'_n) = \delta_{mn}$ (Kronecker delta). Also by the corollary there are numbers $c_i > 0$ such that

$$e_0 = -(c_1 e_1 + \cdots + c_d e_d).$$

Since H_0 is orthogonal to e_0 there is a real number c such that H_0 can be defined as the set of points x satisfying $B(x, e_0) = -c$. Every point y of V can be written as $y = \xi_1 e'_1 + \cdots + \xi_d e'_d$

for some $\xi_i \in \mathbb{R}$. Since C is the intersection of the "positive" half-spaces with respect to its walls, $y \in C$ if and only if $B(y, e_i) > 0$ for $1 \leq i \leq d$ and $B(y, e_0) > -c$ if and only if $\xi_i > 0$ for $1 \leq i \leq d$ and $c_1\xi_1 + \cdots + c_d\xi_d < c$. Since C is non-empty and the c_i's are positive, we must have $c > 0$.

Define $a_m = \frac{c}{c_m} e'_m$ for $1 \leq m \leq d$. Then C consists of the points $\sum_{m=1}^{d} \lambda_m a_m$ with $\lambda_m > 0$ for $1 \leq m \leq d$ and $\lambda_1 + \cdots + \lambda_d < 1$. Hence C is the open simplex with vertices $0, a_1, ..., a_m$. (Adapted from [6, chapter V, section 3, proposition 8]) ■

We motivated the examples of finite reflection groups given in section 1B by considering the symmetry groups of regular polytopes in $\mathbb{R}^n$. The equivalent for infinite reflection groups would be periodic tilings of $\mathbb{R}^n$ by regular polytopes. By the theorem above, if we make the assumption that our group is irreducible and essential, then the tiles must simplices. In the previous example we discussed $\mathbb{R}$, so now we consider the 2–dimensional case in detail.

Example I.6 (Triangle Groups)**.** Consider a triangle, and the infinite reflection group generated by reflections in its sides. The interior angles must be integer sub-multiples of π in order that the group be discrete. Let them be $\frac{\pi}{l}$, $\frac{\pi}{m}$, and $\frac{\pi}{n}$, such that $l \leq m \leq n$. Necessarily each of these is at least 2. In Euclidean space we know that the angles sum up to π, that is

$$\beta := \frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$$

So the only possible triangles are $(l, m, n) = (2, 3, 6),\ (2, 4, 4),\ \text{or}\ (3, 3, 3)$. These correspond to the tiling of the plane by regular hexagons, squares, and equilateral triangles respectively, which can be seen by taking the barycentric subdivision of the hexagon and square (see appendix B.4, and figure B.4 on page 85). The tiling in the case $(2, 4, 4)$ is illustrated in figure I.6.

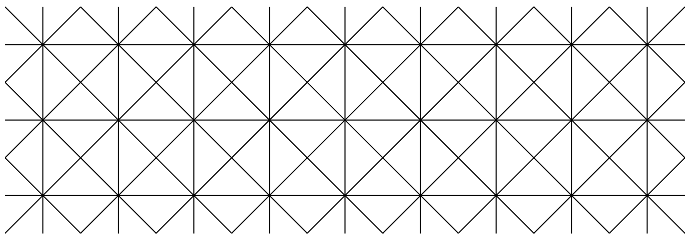


Figure I.6: Periodic tiling of the plane by (2,4,4) triangles, which corresponds to the tiling by squares.

With knowledge of 2–dimensional spherical and hyperbolic geometry we know that we can generalise this. Choose any triple of integers at least 2 (l, m, n) ordered as above, and compute the "angle sum" β. If $\beta > 1$, the corresponding triangle exists on a sphere, and if $\beta < 1$ it exists on the hyperbolic plane[6]. The geometric reflection groups which arise in this way are called **triangle groups**, and are often written $\triangle(l, m, n)$.

As in the Euclidean case, there will only be finitely many triples in the spherical case. These again will make up the barycentric subdivision of some regular polygon on the sphere with which it is tiled. Periodic tilings of the sphere by regular polygons correspond exactly to the regular polyhedra (the platonic solids). We have already seen $\triangle(2, 3, 5)$, which corresponds to the dodecahedron, in example I.4. From the discussion about the relation between finite reflection groups and the sphere at the end of the last section, we know that the spherical case actually corresponds to finite reflection groups in 3–dimensions. (Based on [18, chapter 11])

[6]We see such a triangle group in example IV.2.

Notes

1. The material in the first section closely follows the exposition in chapter I of [8], although we have reordered some of the material where it made sense so that the ideas which were directly transferable to the infinite case were all presented first and together, before covering the results particular to the finite case.
2. The material in the second section was sourced more widely, but the initial transfer of material from the first section was arrived at by analysing the assumptions made in [8] about finiteness.
3. Theorems I.2 and I.3 are both given in [6] in statements which say slightly more, in particular about the fixed point sets under the action of reflection groups, however for simplicity we chose to state only the parts concerning the shape of chambers.

Chapter II

Coxeter Groups (Combinatorial Reflection Groups)

This chapter is logically independent from the first chapter with the exception of the very last page (which could be made independent), and will seem like an abrupt change of gear. The aim will be to unify the two sets of ideas in chapter III. We first take a broad and more philosophical look at combinatorial group theory in general. Following this however we shall formally introduce Coxeter groups as combinatorial groups, as well as the language we use to talk about them. The main body of the chapter is devoted to proving some combinatorial properties of Coxeter groups surrounding a notion of length which we define on the group. We shall end the chapter by proving the equivalence of Coxeter groups and combinatorial reflection groups, thus justifying the title above.

II.1 The Problems with Combinatorial Group Theory

Some groups arise very naturally in many contexts, and the contexts in which they arise give us new tools to study and understand those groups. Combinatorial group theory abstracts away from this in the hope of studying groups more generally, and finding groups which have not been observed in the wild. Combinatorial groups are defined via group presentations, there are different ways to think about group presentations: as quotients of free groups, as a monoid with a certain equivalence relation, or, as is perhaps the easiest, in terms of generators and relations.

Definition II.1. Let $\mathscr{A}$ be an abstract set whose elements we shall call **generators**. The elements of $\mathscr{A}$, along with the elements of $\mathscr{A}^{-1} := \{a^{-1} \mid a \in \mathscr{A}\}$ (this is another abstract set, in bijective correspondence with $\mathscr{A}$, think of the elements of $\mathscr{A}$ as names, then the elements of $\mathscr{A}^{-1}$ are their formal inverses), will be called **letters**, and the set of all letters we shall call an **alphabet**. A finite string of letters will be called a **word**. Let $\mathscr{R}$ be a set of expressions of the form "word in the alphabet"="word in the alphabet", each expression in $\mathscr{R}$ will be called a **relation**. The a **group presentation** is of the form $\langle \mathscr{A} \mid \mathscr{R} \rangle$. $\mathscr{A}$ is called the **set of generators**, and $\mathscr{R}$ the **set of relations**.

A group presentation is a group, with operation the concatenation of words, followed by deletion of any occurrences of aa^{-1} or $a^{-1}a$, for $a \in \mathscr{A}$:

$$abc^{-1} * ccbab = abc^{-1}ccbab = abcbab$$

the identity is the empty word, denoted ε, and associativity and the existence of inverses are easy to prove. Two words represent the same group element if one can be transformed into the other by a series of substitutions using the relations, or equalities which can be derived from the relations.

Notation II.1. In a combinatorial group there is a subtle but important difference between a group element and a *word representing* a group element. Often one can get away with not worrying about the distinction, however we shall need to prove some results which talk about this explicitly, and so it will be necessary to distinguish them in our notation. We shall write a word as a sequence $\mathbf{a} = (a_1, ..., a_k)$, for $a_i \in \mathcal{A} \cup \mathcal{A}^{-1}$, and then the group element it represents would be written $a_1 \cdots a_k$. Note that in a given group, one could have that $a_1 \cdots a_k = a'_1 \cdots a'_{k'}$, but never that $(a_1, ..., a_k) = (a'_1, ..., a'_{k'})$ unless $k = k'$ and $a_i = a'_i$ for all i. This follows the conventions in [6].

Definition II.2. $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is a **presentation** of a group G if they are isomorphic as groups. A group presentation is finite if $\mathcal{A}$ and $\mathcal{R}$ are finite sets; G is called **finitely presented** if it has a finite group presentation (being finitely presented does not mean that G itself is finite, for example, the group of integers has presentation $\langle 1 \mid \rangle$, which is almost as simple a presentation as one could write down, though there are infinitely many integers). We shall only concern ourselves with finitely presented groups here.

At first sight group presentations seem to be a marvellous way to describe groups. They get right to the heart of the group structure, listing only the generators and a few choice relations, instead of having to list every element, and specify how the group operation works for each pair of elements. It is then very easy to prove properties of every element in the group, because one merely needs to show the generators satisfy the properties, and that they are consistent with the relations.

Example II.1. To illustrate the efficiency of presentations in writing down groups, consider the monster group M, of order $\sim 8 \times 10^{53}$, which nevertheless has presentation

$$\langle a, b, u \mid a^2 = b^3 = (ab)^{29} = u^{50} = (au^{25})^5 = (ab(b^2a)^5 b(ab)^5 b)^{34} = \varepsilon, u = (ab)^4(abb)^2 \rangle$$

and so only requires 2 generators, since the last relation gives u in terms of a and b [11, pp. 228–234].

What then is the problem with combinatorial group theory? Well, these advantages really come into their own when a group presentation is being used descriptively, that is if one is writing down a known group. What if, however, we chose a random set of letters for our generators, and a random set of relations. What could we say about that group? Could we find a naturally occurring group to which it is isomorphic? Given another random group presentation, would one know whether they define the same group or not, and is there a fast way to decide this in general? Could we even check whether the group we originally wrote down was anything other than the trivial group in disguise? Moreover, we can see from the definition that we can transform one word into another in a given group using the relations, so two different words can correspond to the same group element; is there a way to decide whether, given two arbitrary words, this is the case or not?

The answer to all of these questions is no in general, and this is the Achilles' heel of combinatorial group theory, in principle one could spend years studying a group presentation, discover hundreds of properties of it, and never know that in fact one was studying the trivial group all along. If, on the other hand, one can show that the combinatorial group is the same as some naturally occurring group, say a group of symmetries of some geometric object, or as a group of matrices, then this problem is solved (the trivial case at least).

Such questions as these were first formally discussed by M. Dehn in his 1912 paper *"Über unendliche diskontinuierliche Gruppen"* [14], wherein he writes:

1. **The identity problem**: An element of the group is given as a product of generators. One is required to give a method whereby it may be decided in a finite number of steps whether this element is the identity or not.

2. **The transformation problem**: Any two elements S and T of the group are given. A method is sought for deciding the question whether S and T can be transformed into each other, i.e. whether there is an element U of the group satisfying the relation
$$S = UTU^{-1}.$$
3. **The isomorphism problem**: Given two groups, one is to decide whether they are isomorphic or not (and further, whether a given correspondence between the generators of one group and elements of the other group is an isomorphism or not).

The first of these problems has subsequently become known as the **Word Problem**, and is stated more generally as deciding whether two words represent the same group element, not just whether they represent the identity. The second problem is now known as the **Conjugacy Problem**, and is equivalent to determining the conjugacy classes of a group. Since the invention of the modern computer, and K. Godël's work on incompleteness [15], these questions haves also been extended to ask, if an algorithm does exist [for a particular class of groups] to decide one of these problems, how fast can that algorithm be made (this is the so-called P vs. NP problem).

It is not without reason that Dehn wrote 'here there are three fundamental problems whose solution is very difficult and which will not be possible without a penetrating study of the subject'. Anyone who has played around with conjugacy classes will know that the second problem is hard enough even if we have a "normal" group, never mind just the group presentation. To illustrate the other two, consider the following examples.

Example II.2.

1. Consider the group presentation
$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ac)^2 = \varepsilon \rangle$$
and the word
$$(b, a, c, b, a, b, c, a, b, c)$$
(This expression is taken from [9, p. 62].) If asked to find an equivalent word with minimal number of letters, one might play about with this and the group relations for hours, and even if one found an equivalent word, of length say 3, how would one know it was minimal? In fact this group is isomorphic to S_4, the permutation group on 4 letters, with isomorphism
$$\begin{cases} a \mapsto (12) \\ b \mapsto (23) \\ c \mapsto (34) \end{cases}$$
If one then explicitly wrote out the element above as a product of transpositions, it is the work of a few seconds to verify that this word in fact represents the identity.
2. Consider the group presentation [23, p. 1]
$$\langle a, b \mid a^{-1}ba = b^2, ab = a^2 \rangle$$
It turns out that this is in fact the trivial group in disguise, which can be seen with a little

bit of manipulation of the relations:

$$\begin{aligned} ab = a^2 &\Rightarrow b = a \\ &\Rightarrow a^{-1}ba = b^{-1}bb = b^2 \\ &\Rightarrow b = b^2 \\ &\Rightarrow a = b = \varepsilon \end{aligned}$$

This particular example is quite straightforward; but harder is writing down a procedure which will work for any presentation.

In fact there exist groups for which the Word Problem is not only very difficult, but provably unsolvable, though examples tend to be constructed and rather artificial as a result [10, section 9.4, proposition 19]. Similarly the Conjugacy and Isomorphism Problems are insoluble in general; see [10, chapter 9], proposition 19, section 9.5, paragraph 1; and theorem 25 respectively.

There is one more serious problem with defining a group purely using a presentation, which will be a constant concern for us when looking at Coxeter groups, and which is a direct result of the Isomorphism Problem: a given group will have many different presentations, with different relations, and even different numbers of generators. For abelian groups, one can use the equivalence with $\mathbb{Z}$-modules and then the linear structure on these, to write down a procedure to reduce a given set of generators and relations to a "most efficient" presentation [21, chapters 9–12]. In general however one cannot do this, so one can only work with the presentation one is given. Then the properties one proves are, in general, dependent on the choice of presentation, and so not necessarily fundamental properties of the group itself. In particular in our case, there will be a number of very important constructions (the Coxeter and Davis complex, as well as the reflection representation) which are entirely dependent on the choice of presentation[1].

II.2 Combinatorial Reflection Groups

We return now, with this in mind, to reflection groups; but in contrast to the first chapter we shall consider a class of purely combinatorial groups which deserve the name combinatorial reflection groups. The study of these groups was initiated by J. Tits in the 1960s, who called them Coxeter groups after H. S. M. Coxeter; K. Brown cites [26], though as it is an unpublished manuscript we are unable to verify this. The definition of a Coxeter group is as follows.

2A The Definition

Definition II.3. Let $S = \{s_1, ..., s_n\}$ be a set of generators, then the group $W = \langle s_1, ..., s_n \mid (s_i s_j)^{m_{ij}} = \varepsilon \rangle$, where

$$\begin{cases} m_{ij} = 1 & \text{if } i = j, \\ 2 \le m_{ij} \le \infty & \text{if } i \ne j. \end{cases}$$

is a **Coxeter group**, and by extension every group which admits a presentation of this form is also a Coxeter group. As a result of the final paragraph of the previous section, it is necessary to keep track not only of the isomorphism type (W) of the Coxeter group in question, but which presentation of that group we are using. Hence we shall almost always refer to a **Coxeter system**, which consists of the pair (W, S), which records this information.

We shall refer to the condition that a group satisfies this definition as **(C)**, which stands for Coxeter.

[1]The way people approach this problem is by studying so-called rigidity, which is mentioned in appendix A.3.

The m_{ij}'s are allowed to be infinite, this just means that there is no relation between the generators s_i and s_j, and by convention, we omit these expressions from the presentation. What, one might justifiably ask, has this definition got to do with reflection groups? Certainly each of the generators is an involution just as a reflection, and it cannot be denied that such presentations are particuly simple, as we have very tight control over what kinds of relations are permitted, but can we really claim to have described every group which might be called a reflection group using this definition? For this we need a clear idear of what it would mean for a combinatorial group to be a reflection group (we follow here the exposition of [8, pp. 33–35] until the next theorem).

Let G be a combinatorial group, with generators $A = \{a_1, ..., a_n\}$, which we want to be "reflections", so naturally our first requirement will be that each generator has order 2 in G. It is natural also to expect that the conjugates of a reflection is also a reflection, since conjugation is like a change of basis which does not fundamentally affect the geometry. We therefore make the following definition.

Definition II.4. A **reflection** in a combinatorial reflection group is any group element conjugate to a generator. For a Coxeter system (W, S), we write $R = \{wsw^{-1} \mid w \in W, s \in S\}$ for the set of reflections.

Being an element of order 2 is not sufficient to characterise a reflection (think of the antipodal map on a sphere for example); reflections have mirrors which we want to account for. Let $\mathcal{H}$ be a set in bijective corresopndence with the reflections in G, via $H \mapsto a_H \in R$ for $H \in \mathcal{H}$. Since all reflections are conjugate to elements of A in G, we can define an action of G on $\mathcal{H}$ via

$$a_{gH} = g a_H g^{-1} \qquad \text{for } g \in G \tag{II.1}$$

that is g maps H to the wall corresponding to the reflection which is obtained from a_H via conjugation by g. This captures most of the picture, but we also need to consider the action of G on the "half-spaces" associated to $\mathcal{H}$. We can think of the half-spaces as elements of the set $\mathcal{H} \times \{\pm 1\}$ (having chosen an orientation for each mirror in some way, you have a "positive" and a "negative" half-space). Then it is sufficient to define the action of the generators only. We need a_i to act on $\mathcal{H} \times \{\pm 1\}$ via

$$\rho_{a_i}(H, \sigma) = \begin{cases} (H, -\sigma) & \text{if } H \text{ corresponds to } a_i, \\ (a_i H, \sigma) & \text{else} \end{cases} ; \text{ where } \sigma \in \{\pm 1\}. \tag{II.2}$$

So given a half-space with respect to a wall H, a_i swaps this to the other half-space if a_i is a reflection in H, but changes to the corresponding half-space of the image of the wall H as given by (II.1). This is sufficient to give a sensible definition of what it means for a group to be a combinatorial reflection group.

Definition II.5. A group G generated by a set A is a **combinatorial reflection group** if there is an action of G on $\mathcal{H} \times \{\pm 1\}$ such that every generator acts via the involution ρ defined above. We shall refer to the the condition of satisfying this definition **(A)** for action, and it says the group must act on a certain set in a certain way.

Theorem II.1. *The definitions of combinatorial reflection groups and Coxeter groups are equivalent.* ■

This says remarkably that Coxeter groups describe all possible groups which might reasonably be called reflection groups (in the sense of the preceding discussion). For the proof we shall follow the scheme outlined in [8, chapter II], although we deviate in the details. It requires proving the equivalence of six conditions on combinatorial groups, the first of which is **(C)**, and the last of which is **(A)**. We shall be in a position to complete the proof by the end of the chapter.

2B Descriptions of Coxeter Groups

Let us look back at the definition of Coxeter systems and see what we can immediately deduce about them. We have already noted that each generator is of order 2, this means that all generators are self-inverses, so in the language of combinatorial group theory, the collections of letters S and S^{-1} coincide for Coxeter groups, so we can justifiably call S the alphabet of our Coxeter group (see definition II.1). We might ask what is a necessary and sufficient condition for a Coxeter group to be abelian, since these are a well-known and studied class of groups. Two generators commute if and only if the order of their product is 2, i.e. W is abelian if and only if $m_{ij} = 2$ whenever $i \neq j$, so perhaps abelian Coxeter groups are not the most interesting objects to study. Another easy observation is that the relations are symmetric, that is $m_{ij} = m_{ji}$, indeed:

$$\begin{aligned}(s_i s_j)^{m_{ji}} &= s_i s_j \cdots s_i s_j = s_i (s_j s_i)^{m_{ji}-1} s_j \\ &= s_i (s_j s_i)^{-1} s_j = s_i s_i s_j s_j = \varepsilon.\end{aligned}$$

This shows that m_{ij} divides m_{ji}, but applying the same argument to $s_j s_i$ reduces to a contradiction unless they are equal.

The very notation which we have used suggests another definition we could make, which although at first is little more than a typographical convenience, will be be instrumental later (see definition III.1).

Definition II.6. Given a Coxeter system (W, S), the associated **Coxeter matrix** $M = (m_{ij})_{i,j=1}^{\#S}$ is the square matrix of dimension $\#S$, whose ij^{th} entry is m_{ij}.

From the definition of (W, S) and the observations above, we know that M is symmetric with 1's along the leading diagonal, and all other entries in $\{2, ..., \infty\}$. It is clear that specifying a matrix of this form is equivalent to specifying a Coxeter system. Another way of expressing a Coxeter system is as follows.

Definition II.7. Let (W, S) be a Coxeter system with Coxeter matrix $M = (m_{ij})_{i,j}$. The **Coxeter diagram** $\mathcal{V}$ of (W, S) is a labelled graph whose vertex set is S, and where the edge $\{s_i s_j\}$ is present and labelled m_{ij} whenever $m_{ij} \geq 3$. By convention, edges labelled 3 usually have their label suppressed.

Specifying a Coxeter diagram uniquely specifies a Coxeter system and its Coxeter matrix.

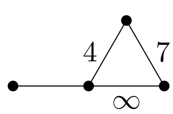
Example II.3. The following are 3 equivalent representations of the same Coxeter system. Group presentation:

$$\langle s_1, ..., s_4 \mid s_i^2 = (s_1 s_2)^3 = (s_1 s_3)^2 = (s_1 s_4)^2 = (s_2 s_4)^4 = (s_3 s_4)^7 = \varepsilon \rangle$$

Matrix:

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & \infty & 4 \\ 2 & \infty & 1 & 7 \\ 2 & 4 & 7 & 1 \end{pmatrix}$$

Diagram:



Once one gets used to the notation, the Coxeter diagram tends to be the easiest and most efficient representation.

Why do we omit the edges of $\mathcal{V}$ labelled 2? It certainly simplifies the diagram significantly, otherwise each graph would be the complete graph on n vertices, with labels, which would be inscrutable; there is however a more mathematical reason, which relates to the fact that $m_{ij} = 2$ means that s_i and s_j commute, so do not really interact with one another. This shall be made more precise in the next section.

Example II.4. We consider Coxeter systems with small numbers of generators.

1. If $S = \emptyset$, then W must be the trivial group.
2. If there is only one generator s, $W = \langle s \mid s^2 = \varepsilon \rangle$ is the cyclic group of order 2.
3. If $S = \{s, s'\}$, we get one interesting relation $W = \langle s, s' \mid s^2 = s'^2 = (ss')^m = \varepsilon \rangle$ where $2 \leq m \leq \infty$. So m parametrises an infinite family of groups which are typically denoted $I_2(m)$. We distinguish two cases: m finite and m infinite.

   If m is finite, W is the dihedral group of finite order $2m$ with which everyone is familiar. W can be made to act on $\mathbb{R}^2$ by letting s and s' act as reflections in two lines through the origin which intersect act an angle $\frac{\pi}{m}$. Another common presentation for this group is

   $$\langle s, p \mid s^2 = p^m = \varepsilon \rangle$$

   where p corresponds to the rotation ss'. Using the equation $p^n s = sp^{m-n}$ which holds for any integer n, one can take any word in $\{s, p\}$ and write it in the form p^a or sp^a for some $0 \leq a \leq m-1$, which in the alphabet $\{s, s'\}$ reads $(s, s', ..., s, s')$ or $(s', s, ..., s, s')$ respectively. Since s and s' do not commute, these are all distinct elements except for the longest of these words which correspond to the same element, since

   $$(ss')^m = \varepsilon \Longrightarrow \overbrace{ss' \cdots}^{\text{lenght } m} = \overbrace{s's \cdots}^{\text{lenght } m}.$$

   Explicitly for $W = I_2(4) \cong D_4$, we have

   $$\langle s, s' \mid s^2 = s'^2 = (ss')^4 = \varepsilon \rangle$$

   which has 8 elements:

   $$\begin{array}{ccc} & \varepsilon & \\ s & & s' \\ ss' & & s's \\ ss's & & s'ss' \\ & ss'ss' = s'ss's & \end{array}$$

   If m is infinite there is no extra relation, so all of the words $(s, s', ...)$ and $(s', s, ...)$ represent distinct elements, they are in fact the unique minimal expressions (see definition II.10 below) for each of the elements. This group is called the infinite dihedral group, written D_∞, and it is an example of an infinite Coxeter group, see example I.5. (The above is based on [6, chapter IV, section 1.2])

   The above discussion in fact outlines a solution to the Word Problem for dihedral groups: given a word, use $p^n s = sp^{m-n}$ in the case m is finite to get it into the standard form, after deleting occurrences of ss and $s's'$. Then this is unique unless the standard form has length m, in which case there are two possibilities. For the infinite case just delete occurrences of ss and $s's'$ to get an alternating word which is unique to that group element. Now all one needs to do in either case is compare these standard forms.

Example II.5. Every example we have done so far has been motivated in some way or another by geometry. There is an example completely motivated by algebra. Consider the group of permutations of n objects S_n. It is an elementary result about permutations that every permutation can be decomposed into a product of transpositions, which is another way of saying that S_n is generated by transpositions, which are permutations of order 2. This suggests that S_n might be a Coxeter group. S_n is generated by

$$S = \{(1,2),(2,3),...,(n-1,n)\}$$

If we denote the transposition $(i, i+1)$ by s_i for $1 \leq i < n$ then it is easy to see that the order m_{ij} of $s_i s_j$ is

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i - j = \pm 1, \\ 2 & \text{otherwise.} \end{cases}$$

The group generated by S with the corresponding Coxeter relations turns out to by isomorphic to S_n, and has Coxeter diagram as seen in figure II.1. In the context of Coxeter groups, these groups are denoted A_{n-1}[2] (where the subscript denotes the number of generators). We already saw a foretaste of this in (1) of example II.2. With this example, Cayley's theorem says that every finite group is a subgroup of a Coxeter group.

Figure II.1: The Coxeter diagram corresponding to the group A_{n-1}, which has $n-1$ nodes.

2C Special Subgroups and Irreducibility

Definition II.8. Let (W, S) be a Coxeter system, and let T be a subset of S. We write W_T for the subgroup $\langle T \rangle$ of W generated by T (i.e. the group generated by T with all the relations of W which use only letters of T). Such subgroups are called **special subgroups**, and then T is called a **special subset** of S.

It is clear that it does not make sense to talk about the special subgroups of a Coxeter *group*, only of a Coxeter *system*, because the definition is entirely dependant on the choice of presentation. Note also that in general a Coxeter system will have many subgroups which are not special, as an example, if the Coxeter group can be realised as a geometric reflection group in the sense of chapter I, then we might consider its subgroup of orientation preserving symmetries which necessarily contains no reflections at all, so certainly cannot be generated by them.

Proposition II.1. *Let (W, S) be a Coxeter system, with special subgroup W_T, then (W_T, T) is a Coxeter system. [6, chapter IV, section 1, theorem 2(i)]*

Proof. After relabelling, let $S = \{s_1, ..., s_n\}$, and $T = \{s_1, ..., s_k\}$ for some $k \leq n$. If the Coxeter matrix corresponding to (W, S) is $M = (m_{ij})_{i,j=1}^n$, from the definition all of the relations defining W_T are given by the matrix $\overline{M} = (m_{ij})_{i,j=1}^k$, which is a valid Coxeter matrix, and so defines a Coxeter system. Hence (W_T, T) is a Coxeter system. ∎

[2] This notation should not be confused with the alternating subgroup of S_n, which is often denoted A_n. We shall not discuss the alternating subgroup, so there should be not chance for misunderstanding.

Theorem II.2. *Let (W, S) be a Coxeter system, and let $(T_i)_{i \in I}$ be a partition of S for some index set I such that for all $i \neq j \in I$, the generators in T_i commute with those in T_j. Then W is isomorphic to the direct product of the collection $(W_{T_i})_{i \in I}$.*

Proof. We proceed by induction on the size of I. There is nothing to prove if $\#I = 1$. Since each set of generators T_i commutes with every other set, W_{T_i} commutes with $\bigcup_{j \in I \setminus \{i\}} W_{T_j}$, so W_{T_i} is normal in W. The product of all W_{T_i}'s contains all of S, and hence W. By induction $W_{S \setminus T_i}$ is the direct product of $(W_{T_j})_{j \in I \setminus \{i\}}$, and so it only remains to show that W_{T_i} intersects with this trivially, but this follows from the fact that the T_i's partition S. [17, proposition 2.2] ■

This formalises the assertion that the generators commuting means that they do not interact. It also motivates the following definition.

Definition II.9. A Coxeter system is **irreducible** if it does not admit a non-trivial direct product decomposition as in the above theorem. Otherwise it is called **reducible**.

We can simplify our study of Coxeter systems because we only need to consider irreducible Coxeter systems; all others can be built out of these. This also fully justifies us omitting the edges labelled 2 in the Coxeter diagram:

Lemma II.1. *A Coxeter system is irreducible if and only if the corresponding Coxeter diagram is connected. If the diagram is not connected, the connected components correspond to the irreducible components of the direct product decomposition. [loc. cit.]* ■

II.3 Some Combinatorial Results

In this section we shall get our hands dirty so to speak, and prove some combinatorial results about Coxeter systems. These are just a few of the results which one can prove, but we have chosen results which we think are quite interesting, and not too hard to formulate; moreover they will turn out to be very useful later. For those interested in group theory of this flavour, see [6] in particular chapter IV section 1; and [9] chapters 2–5, who explore the combinatorial properties of Coxeter groups more deeply. First a definition which is indispensable: one can define a notion of length on a combinatorial group.

3A The Length Function

Notation II.2. When referring to the indexed generators in the set S, we have typically been labelling them $s_1, ..., s_n$, where naturally each is distinct from the others. As we start to write down expressions for particular group elements of W, in which the same generator may appear multiple times in different places, we shall avoid using the same notation to limit confusion. We shall tend to use $t_1, ..., t_k$, where each t_i is understood to be a generator in S, but different t_i's may correspond to the same generator.

Definition II.10. Let (W, S) be a Coxeter system, and let $w \in W$. If $(t_1, ..., t_d)$ is a word representing w in the generators S, its length is the integer d. A word representing w of minimal length is called a **reduced** word for w, or a **minimal expression**. Then the **length** of w, written $l_S(w)$, or just $l(w)$ if S is clear from context, is the length of a reduced word for w.

As with special subgroups, the length of group elements is dependent on the choice of presentation, and hence on S, which is why we record this information in the notation of the function. We shall prove three propositions using the length function:

Proposition A. *Let (W, S) be a Coxeter system, and let $w \in W$. Then $l(sw) = l(w) \pm 1$ for all $s \in S$. [9, lemma 2.2.1]*

Proposition B. *Let (W, S) be a Coxeter system with length function l. W is finite if and only if there is a unique longest element in W with respect to l. [9, theorem 5.2.4]*

Proposition C. *A group W generated by a set S is a Coxeter group if and only if it satisfies the deletion condition* ***(D)****:*

> *If $w \in W$ is represented by the word $(t_1, ..., t_d)$ with $d > l(w)$, then there are indices $i < j$ such that $w = t_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_d$.*

where $\hat{t}_i$ indicates that that letter has been deleted from the expression. [8, section II.1, corollary]

First we shall need some basic properties of the length function.

Lemma II.2. *Let (W, S) be a Coxeter system, and let $w, w' \in W$.*

1. *$l(w) = 0$ if and only if $w = \varepsilon$,*
2. *$l(w) = 1$ if and only if $w \in S$,*
3. *$l(w^{-1}) = l(w)$,*
4. *$\|l(w) - l(w')\| \leq l(ww') \leq l(w) + l(w')$.*

[3, proposition 1.1]

Proof. The first two claims are obvious.

3. Suppose $(t_1, ..., t_d)$ is a minimal expression for w, but that $l(w^{-1}) > d$. This gives a contradiction since $(t_d^{-1}, ..., t_1^{-1})$ is a word representing w^{-1} of length d, so $l(w^{-1}) \leq l(w)$. A symmetric argument gives the opposite inequality.
4. Clearly we have that $l(ww') \leq l(w) + l(w')$. On the other hand

$$l(w) = l((ww')w'^{-1}) \leq l(w'w) + l(w'^{-1}) \overset{3}{=} l(ww') + l(w')$$

hence $l(w) - l(w') \leq l(ww')$.

■

Proof of proposition A. By an application of lemma II.2(4) with one of the group elements replaced by s, we get $l(w) - 1 \leq l(sw) \leq l(w) + 1$, so we need only rule out the possibility that $l(sw) = l(w)$. We shall do this by showing that $l(sw)$ and $l(w)$ have different parities. Let $F(S)$ be the free group generated by set S, then let $\phi : F(S) \mapsto \{\pm 1\}$ be such that $s \mapsto -1$ for all $s \in S$. This extends uniquely to a well-defined homomorphism on $F(S)$. W is obtained from $F(S)$ by taking the quotient by the subgroup generated by the elements $(s_i s_j)^{m_{ij}}$, since the relations of W are of the form $(s_i s_j)^{m_{ij}} = \varepsilon$. Clearly all of these relations lie in the kernel of ϕ, so it descends to a well-defined homomorphism $\overline{\phi} : W \mapsto \{\pm 1\}$. Now let $(t_1, ..., t_{l(w)})$ be a minimal expression for w, where $t_i \in S$ for all i;

$$\overline{\phi}(w) = \overline{\phi}(t_1) \cdots \overline{\phi}(t_{l(w)}) = (-1)^{l(w)},$$

and

$$\overline{\phi}(sw) = \overline{\phi}(s)\overline{\phi}(w) = (-1)^{l(w)+1}.$$

$\overline{\phi}(w)$ measures the parity of $l(w)$, so the parity of $l(w)$ and $l(sw)$ differ; in particular $l(w) \neq l(sw)$. [17, proposition 5.2] ■

Remark II.1. This proof can immediately be generalised to the case were s is replaced by any reflection $r \in R$, to say that $l(rw) \neq l(w)$ for all $w \in W$, using the observation that the obvious word representing r has odd length.

3B Proposition B

We are going to need to work quite a bit harder to prove the second proposition. For the proof we need a lemma (lemma II.5), but we can in fact prove a more general result by proving the "opposite" to this lemma (lemma II.4) as well. The proof of this other lemma will in fact be useful to us later, and uses some of our ideas from section 2A, so will be very worth while. First of all some definitions.

Definition II.11. Let $\mathbf{t} = (t_1, ..., t_k)$ be a word in the alphabet S, we write

$$\mathbf{p}_i := (t_1, ..., t_{i-1}, t_i, t_{i-1}, ..., t_1);\text{ for } 1 \leq i \leq k$$

and then the tuple of these $\mathbf{p}_i$'s[3] is

$$\widehat{R}(\mathbf{t}) := (\mathbf{p}_1, \ldots, \mathbf{p}_k).$$

This defines a collection of reflections derived from a word $\mathbf{t}$. Given a reflection $r \in R$, we define $n(\mathbf{t}; r)$ to be the number of $\mathbf{p}_i$'s in $\widehat{R}$ which are words representing r. Then set

$$\eta(\mathbf{t}; r) := (-1)^{n(\mathbf{t};r)}$$

Lemma II.3. *Let (W, S) be a Coxeter system, and let $w \in W$. If $\boldsymbol{t} = (t_1, ..., t_k)$ and $\boldsymbol{t}' = (t'_1, ..., t'_{k'})$ are two words representing w, then*

$$\eta(\boldsymbol{t}; r) = \eta(\boldsymbol{t}'; r)$$

for all $r \in R$. Hence we can define

$$\eta(w; r) := (-1)^{n((t''_1,...,t''_k);r)}$$

where $(t''_1, ..., t''_k)$ is an arbitrary expression for w.

Proof. We omit the proof so this section does not get too long, it may be found in [4], see (1.17) on p. 14. ■

We shall go back to our action ρ defined on the "half-spaces" in equation (II.2). Since we said that $\mathcal{H}$ was in bijective correspondence with the set of reflections R, the set of half-spaces is in bijective correspondence with the set $R \times \{\pm 1\}$, and we can define the corresponding action of W on this set using equation (II.1):

$$\pi_s(r, \sigma) = (srs, \sigma\eta(s; t))$$

for $s \in S$, since

$$\eta(s; t) = \begin{cases} -1 & \text{if } s = t, \\ +1 & \text{if } s \neq t. \end{cases}$$

As with ρ, this extends to an action of the whole of W on $R \times \{\pm 1\}$, which it turns out can be written as

$$\pi_w(r, \sigma) = (wrw^{-1}, \sigma\eta(w^{-1}; r)) \tag{II.1}$$

see [4, theorem 1.3.2] for details. This was the reason for introducing the η notation. We can now prove the following.

[3] p is chosen to stand for palindrome.

Theorem II.3 (Strong Exchange Condition)**.** *Let (W, S) be a Coxeter system with $w \in W$. Let $(t_1, ..., t_d)$ be an expression for w, and let $r \in R$. If $l(rw) < l(w)$, then there is some index i such that $rw = t_1 \cdots \hat{t}_i \cdots t_d$.*

Proof. The first stage of the proof is to establish the equivalence of the following two conditions:

1. $l(rw) < l(w)$
2. $\eta(w; r) = -1$.

If we assume that $\eta(w; r) = -1$, and $(t'_1, ..., t'_d)$ is a minimal expression for w, we can conclude from the definition that $n((t'_1, ..., t'_d); r)$ is odd, and hence $r = p_i$ for some i, were p_i is the element represented by the word $\mathbf{p}_i$. Hence

$$l(rw) = l(t'_1 \cdots t'_i \cdots t'_1 t'_1 \cdots t'_d) = l(t'_1 \cdots \hat{t}'_i \cdots t'_d) < l(w)$$

as required.

To prove the other direction, we shall prove the contrapositive, so assume that $\eta(w; r) = 1$, then by (II.1)

$$\begin{aligned}\pi_{(rw)^{-1}}(r, \sigma) &= \pi_{(w)^{-1}} \pi_r(r, \sigma) = \pi_{(w)^{-1}}(r, -\sigma)\\ &= (w^{-1}rw, -\sigma\eta(w; r)) = (w^{-1}rw, -\sigma)\\ \pi_{(rw)^{-1}}(r, \sigma) &= (w^{-1}rrrw, \sigma\eta(rw; r)) = (w^{-1}rw, \sigma\eta(rw; r))\end{aligned}$$

which means that $\eta(rw; r) = -1$, and so by the first part of the proof we conclude that $l(rrw) < l(rw)$, that is $l(rw) > l(w)$.

Finally we apply the second implication proved. Since $l(rw) < l(w)$, $-1 = \eta(w; r) = (-1)^{n((t_1,...,t_k);r)}$, so $n((t_1, ..., t_k); r)$ is odd, and $r = t_1 \cdots t_i \cdots t_1$ for some index i, hence $rw = t_1 \cdots \hat{t}_i \cdots t_k$. [4, theorem 1.4.3] ■

As its name might suggest, this result is very important and useful in the study of Coxeter groups, not just in the proof of proposition B. We shall indeed have recourse to it in the proof of proposition C. It is also one of the six conditions involved in the proof of theorem II.1 — in fact this result is the first step in this proof. One use of the Strong Exchange Condition is in the proof of the following lemma.

Lemma II.4. *Let (W, S) be a Coxeter system with $w, w' \in W$, and $r \in R$, the set of reflections in (W, S). If $l(w) < l(wr)$ and $l(w') < l(rw')$, then $l(ww') < l(wrw')$.*

Proof. Let us assume that $l(ww') > l(wrw')$ and derive a contradiction. If we set $\tilde{r} = wrw^{-1}$ then it clear that $l(wrw') = l(\tilde{r}ww')$, and it is not hard to show that we are safe in neglecting the possibility that $l(ww') = l(wrw')$. Let $(t_1, ..., t_k)$ and $(t'_1, ..., t'_{k'})$ be minimal expressions for w and w', using the assumption that $l(ww') > l(\tilde{r}ww')$ we can apply the strong exchange condition to deduce that either

$$\begin{aligned}\tilde{r}ww' &= t_1 \cdots \hat{t}_i \cdots t_k t'_1 \cdots t'_{k'}, \text{ or}\\ \tilde{r}ww' &= t_1 \cdots t_k t'_1 \cdots \hat{t}'_j \cdots t'_{k'}.\end{aligned}$$

for some indices i and j. In the first case we get

$$l(wr) = l(\tilde{r}w) = l(t_1 \cdots \hat{t}_i \cdots t_k) < l(t_1 \cdots t_k) = l(w)$$

and in the second case we can note that $wrw' = \tilde{r}ww' = wt'_1 \cdots \hat{t}'_j \cdots t'_{k'}$, and hence

$$l(rw') = l(t'_1 \cdots \hat{t}'_j \cdots t'_{k'}) < l(t'_1 \cdots '_{k'}) = l(w')$$

so both cases contradict the assumptions of the lemma. [4, lemma 2.2.10] ■

Here is a very similar result.

Lemma II.5. *Let (W, S) be a Coxeter system with $w, w' \in W$, and $r \in R$, the set of reflections in (W, S). If $l(w) > l(rw)$ and $l(w') > l(rw')$, then $l(w^{-1}w') < l(w^{-1}rw')$.* ■

The proof of this uses root systems which we have not introduced, so it has been omitted. It can be found in [9, corollary 5.2.2].

Remark II.2. This result is on the face of it quite counter intuitive: we have a reflection r which shortens both w and w', but *lengthens* ww'. One way to resolve this is to imagine that there may be some cancellation between w and w', and by putting r in between them, we keep them apart, and so stopping this cancellation from happening.

Let us play about with the form of this second lemma and see is we can get it to look more similar the first lemma. First note that

$$l(rw) = l((rw)^{-1}) = l(w^{-1}r^{-1}) = l(w^{-1}r)$$

Together with the fact that $l(w) = l(w^{-1})$, the first hypothesis becomes $l(w^{-1}) > l(w^{-1}r)$. Then substituting $w \leftrightarrow w^{-1}$ throughout, the statement becomes

$l(wr) < l(w)$ and $l(rw') < l(w')$ implies $l(ww') < l(wrw')$.

While the lemma which we proved reads

$l(wr) > l(w)$ and $l(rw') > l(w')$ implies $l(ww') < l(wrw')$.

So taken together, we get the result that $l(ww') < l(wrw')$ if either r lengthens both w and w', or if r shortens both w and w'.
Now we can prove the second proposition, which we shall recall briefly first.

Proposition B. *Let (W, S) be a Coxeter system with length function l. W is finite if and only if there is a unique longest element in W with respect to l.*

Proof. If W is finite, since words are finite strings of letters, and each element has a finite length, and there are only finitely many of them, there is an element of longest length. We need to show that it is unique.

As usual, let R be the set of reflections in W, and suppose w_0 is an element of longest length. Then we necessarily have that $l(rw_0) < l(w_0)$ for all $r \in R$, since equality is ruled out by remark II.1. We shall show that any element with this property is unique. Suppose $w_1 \in W$ also satisfies $l(rw_1) < l(w_1)$ for all $r \in R$; then by lemma II.5 we have $l(w_0^{-1}w_1) < l(w_0^{-1}rw_1)$ for any r. Replacing r in this inequality by $w_0rw_0^{-1} \in R$ we get that $l(w_0^{-1}w_1) < l(r(w_0^{-1}w_1))$ for all r. This implies that $w_0^{-1}w_1 = \varepsilon$, i.e. $w_0 = w_1$, since if $w_0^{-1}w_1 \neq \varepsilon$, choose a minimal expression for $w_0^{-1}w_1$: $(t_1, ..., t_d)$. Applying the inequality with $r = t_1$ we get

$$d = l(w_0^{-1}w_1) < l(t_1t_1 \cdots t_d) = l(t_2 \cdots t_d) \leq d - 1$$

a contradiction. Hence w_0 is unique.

The reverse direction is trivial, since we have a running assumption that all our groups are finitely presented. (Adapted from [9, theorem 5.2.4]) ■

Now compare this result to the discussion of finite dihedral groups in example II.4. We are left with the third proposition:

3C Proposition C

Proposition C. *A group W generated by a set S is a Coxeter group if and only if it satisfies the deletion condition* ***(D)****:*

> *If $w \in W$ is represented by the word $(t_1, ..., t_d)$ with $d > l(w)$, then there are indices $i < j$ such that $w = t_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_d$.*

where $\hat{t}_i$ indicates that that letter has been deleted from the expression.

This will in fact do most of the work towards proving theorem II.1. We have met two of the conditions, the deletion condition, and the strong exchange condition, which we shall abbreviate as **(SE)**. We shall be using a number of other conditions throughout this discussion, and it seems best to introduce them all at the start as opposed to in an *ad hoc* fashion, so that we can draw a road map of what we are hoping to achieve.

We have a condition **(C)** called the Coxeter condition, which is the condition of being a Coxeter group, i.e. a group satisfies **(C)** if it satisfies the definition of a Coxeter group (see definition II.3). We shall need, in addition to **(SE)**, the exchange condition **(E)**, which reads:

> Let $(t_1, ..., t_d)$ be a minimal expression for $w \in W$, and let $s \in S$. If $l(sw) = l(w) - 1$, then there is some index i such that $sw = t_1 \cdots \hat{t}_i \cdots t_d$.

Theorem II.3 in fact reads that **(C)** *implies* **(SE)**, and it is clear that **(E)** is just a special case of **(SE)**, so we have already proved that **(C)** *implies* **(E)**. Our plan now will be to prove **(E)** *implies* **(D)**, and then work our way back, proving **(D)** *implies* **(E)** and **(E)** *implies* **(C)**. For the first implication, we shall actually go via yet another condition, called the folding condition **(F)**:

> Let $w \in W$ and $s, s' \in S$ with $l(sw) = l(w) + 1$ and $l(ws') = l(w) + 1$. Then either $l(sws') = l(w) + 2$, or $sws' = w$.

$$\begin{tikzcd} & \textbf{(SE)} \arrow[dr, Rightarrow] & & \textbf{(F)} \arrow[dr, Rightarrow] & \\ \textbf{(C)} \arrow[ur, Rightarrow] & & \textbf{(E)} \arrow[ll, Rightarrow] \arrow[ur, Rightarrow] & & \textbf{(D)} \arrow[ll, Rightarrow] \end{tikzcd}$$

Figure II.2: The plan to prove **(C)** $\Longleftrightarrow$ **(D)**, working clockwise from **(C)**.

Proposition II.2. *If a group W generated by S satisfies* ***(E)****, then it satisfies* ***(D)****.*

Proof. First we prove **(F)**. Suppose $(t_1, ..., t_d)$ is a minimal expression for w, and suppose $l(sws') \neq l(w) + 2$, that is $l(sws') \neq l(ws') + 1$, so $l(sws') < l(ws') + 1$. Then by **(E)** either there is an index i such that $sws' = t_1 \cdots \hat{t}_i \cdots t_d s'$, or $sws' = t_1 \cdots t_d$. In the first case

$$l(sw) = l(sws's') = l(t_1 \cdots \hat{t}_i \cdots t_d s's') = l(t_1 \cdots \hat{t}_i \cdots t_d) = l(w) - 1$$

which contradicts the assumption that $l(sw) = l(w) + 1$. In the second case $sws' = w$ as required, proving **(F)**.

Now we assume **(F)** and prove **(D)**. Suppose $w = t_1 \cdots t_k$ with $k > l(w)$. Necessarily k is at least 2; if $k = 2$, $w = t_1 t_2$ and $l(w) = 0$, i.e. $w = \varepsilon$, so $t_1 = t_2$, and $w = \hat{t}_1 \hat{t}_2$. We now proceed by induction on k. If $t_2 \cdots t_k$ or $t_1 \cdots t_{k-1}$ have length less than $k - 1$ the inductive hypothesis means that $t_2 \cdots t_k = t_2 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_k$ for example, so $w = t_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_k$ as required. Suppose therefore that they both have length $k - 1$, and set $w' = t_2 \cdots t_{k-1}$. Then $l(t_1 w') = l(w) + 1 = l(w' t_k)$ and $l(t_1 w' t_k) = l(w) < l(w') + 2$, so by **(F)** $w = t_1 w' t_k = w'$, or in other words $w = \hat{t}_1 t_2 \cdots t_{k-1} \hat{t}_k$, proving **(D)**. [8, section 3A] ■

This proves one direction of proposition C.

Proposition II.3. *If a group W generated by S satisfies* ***(D)*** *then it satisfies* ***(E)****.*

Proof. Let $(t_1, ..., t_d)$ be a minimal expression for $w \in W$, and let $s \in S$. If $l(sw) = l(w) - 1 < d+1$, we can apply **(D)** to $st_1 \cdots t_d$, and delete 2 of its letters. If one of them is s then we are done, so assume that neither is s and derive a contradiction. We get $sw = st_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_d$, and hence

$$w = ssw = sst_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_d = t_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_d$$

which has length $d-2$, however we assumed that we started with a minimal expression for w of length d, a contradiction. [8, section 3A] ■

As with the proof that **(C)** *implies* **(SE)**, the proof that **(E)** *implies* **(C)** is much harder than the other proofs. The proof builds up sequentially from a lemma to a proposition, and then the theorem; however the difficulty, subtly, and beauty are in inverse proportion to the grandeur of their designation. As such, the lemma in particular needs patience and time to understand and appreciate, so we suggest assuming the proposition and working through the proof of the theorem, thus motivating the proposition; then assuming the lemma and working through the proof of the proposition, which in turn will motivate the lemma. For this proof we follow [6]; notation has been changed from the original only so as to make it consistent with the other material herein, and the phrasing has been altered, or sentences added *only* where convenient to make the style consistent and to explicate certain points. Throughout we assume that W is a group generated by S which satisfies **(E)**. We shall only have recourse to **(E)** in the proof of the lemma.

Lemma II.6. *Let $w \in W$ have length $d \geq 1$, and let D_w be the set of minimal expressions for w, and F_w a map from D_w to a group[4] G. Let $\boldsymbol{t} = (t_1, ..., t_d)$ and $\boldsymbol{t'} = (t'_1, ..., t'_d)$ be in D_w, and suppose that either of*

a) *$t_1 = t'_1$ or $t_d = t'_d$; or*

b) *there are $s, s' \in S$ such that $t_j = t'_k = s$ and $t_k = t'_j = s'$ for j running over all odd indices, and k running over all even indices,*

imply that $F_w(\boldsymbol{t}) = F_w(\boldsymbol{t'})$. Then one may conclude that F_w is constant.

Proof. We proceed in two steps.

Step 1: let $\mathbf{t}$ and $\mathbf{t'}$ be in D_w but assume that $F_w(\mathbf{t}) \neq F_w(\mathbf{t'})$. $w = t'_1 \cdots t'_d$ so $t'_1 w = t'_2 \cdots t'_d$ and is of length at most $d-1$. By **(E)** there is an index h such that $w = t'_1 t_1 \cdots \hat{t}_h \cdots t_d$, so $\mathbf{u}_h = (t'_1, t_1, ..., \hat{t}_h, ..., t_d)$ is in D_w. Since $\mathbf{t'}$ and $\mathbf{u}_h$ share their first letter, (a) is satisfied, so $F_w(\mathbf{t'}) = F_w(\mathbf{u}_h)$. If $h \neq d$, again by a) $F_w(\mathbf{t}) = F_w(\mathbf{u}_h)$, which contradicts our assumption that $F_w(\mathbf{t}) \neq F_w(\mathbf{t'})$, so we must have $h = d$.

This means that if $F_w(\mathbf{t}) \neq F_w(\mathbf{t'})$, $\mathbf{u}_d = (t'_1, t_1, ..., t_{d-1})$ is in D_w and $F_w(\mathbf{t}) \neq F_w(\mathbf{u}_d)$. Starting with this last expression as an assumption we could play the same trick to get $(t_1, t'_1, t_1, ..., t'_{d-2})$ in D_w with $F_w((t_1, t'_1, t_1, ..., t'_{d-2})) \neq F_w((t'_1, t_1, ..., t'_{d-1}))$, and so on.

Step 2: let $(t_1, ..., t_d)$ and $(t'_1, ..., t'_d)$ be in D_w. We define a sequence of words $\{\boldsymbol{\tau}_n\}$ for each

[4] We choose G to be a group because that is what we need for the theorem, in fact this result holds for G a *set*.

integer n with $0 \leq n \leq d-1$, and each of length d as follows

$$\begin{aligned}
\boldsymbol{\tau}_0 &= (t'_1, ..., t'_d), \\
\boldsymbol{\tau}_1 &= (t_1, ..., t_d), \\
&\vdots \\
\boldsymbol{\tau}_{d+1-q} &= \begin{cases} (t_1, t'_1, ..., t_1, t'_1, t_1, t_2, ..., t_q) & \text{if } d-q \text{ is even, } 0 \leq q \leq d, \\ (t'_1, t_1, t'_1, ..., t_1, t'_1, t_1, t_2, ..., t_q) & \text{if } d-q \text{ is odd, } 0 \leq q \leq d. \end{cases}
\end{aligned}$$

Denote by (H_n) the proposition

"$\boldsymbol{\tau}_n, \boldsymbol{\tau}_{n+1} \in D_w$ and $F_w(\boldsymbol{\tau}_n) \neq F_w(\boldsymbol{\tau}_{n+1})$"

Step 1 shows that $(H_n) \Rightarrow (H_{n+1})$ for $0 \leq n < d$, but (b) says precisely that (H_d) is not true, so we get

$$\neg(H_d) \Rightarrow \neg(H_{d-1}) \Rightarrow \cdots \Rightarrow \neg(H_1) \Rightarrow \neg(H_0)$$

Were $\neg$ is the logical negation of a proposition. Hence (H_0) is not true. Since $\boldsymbol{\tau}_0 = (t'_1, ..., t'_d)$ and $\boldsymbol{\tau}_1 = (t_1, ..., t_d)$, it follows that $F_w((t_1, ..., t_d)) = F_w((t'_1, ..., t'_d))$, so F_w is constant. [6, chapter IV, section 1, lemma 4] ■

Remark II.3. This proof can best be understood by drawing an analogy to dominoes. In step 1 we build a machine using hypothesis (a) and **(E)**, which, given a collection of dominoes, will line them up. In step 2 we make the dominoes themselves (the propositions (H_n)). Step 1 then lines them up, and hypothesis (b) topples the last domino. By the way they were built, then the toppling of first domino establishes the lemma.

Proposition II.4. *Let G be a group and $f : S \mapsto G$. For $s, s' \in S$, let m be the order of ss' and define*

$$a(s, s') = \begin{cases} (f(s)f(s'))^{\frac{m}{2}} & \text{if } m \text{ is even,} \\ (f(s)f(s'))^{\frac{m-1}{2}} f(s) & \text{if } m \text{ is odd,} \\ 1 & \text{if } m = \infty. \end{cases}$$

If $a(s, s') = a(s', s)$ whenever $s \neq s'$, there exists a map $g : W \mapsto G$ such that

$$g(w) = f(t_1) \cdots f(t_d)$$

for all $w \in W$ and for any minimal expression $(t_1, ..., t_d)$ of w.

Proof. For any $w \in W$ let D_w be the set of minimal expressions for w, and let $F_w : D_w \mapsto G$ defined by

$$F_w((t_1, ..., t_d)) = f(t_1) \cdots f(t_d).$$

We shall prove by induction on d that F_w is constant, which will establish the proposition. If $l(w) = 0, 1$ then this is trivially the case, since $\#D_w = 1$ in both cases, so assume $d \geq 2$. Let w be of length d, and suppose $\mathbf{t} = (t_1, ..., t_d)$ and $\mathbf{t}' = (t'_1, ..., t'_d)$ are in D_w. By lemma II.6, we need only prove that hypotheses a) and b) imply that $F_w(\mathbf{t}) = F_w(\mathbf{t}')$.

a) We have
$$F_w((t_1, ..., t_d)) = f(t_1)F_{w'}((t_2, ..., t_d)) = F_{w''}((t_1, ..., t_{d-1}))f(t_d)$$
for $w' = t_2 \cdots t_d$ and $w'' = t_1 \cdots t_{d-1}$. So if $t_1 = t'_1$ or $t_d = t'_d$, the induction hypothesis gives $F_w(\mathbf{t}) = F_w(\mathbf{t}')$.

b) Suppose there are $s, s' \in S$ such that $t_j = t'_k = s$ and $t_k = t'_j = s'$ for j running over all odd indices, and k running over all even indices. If $s = s'$ then necessarily $d = 0, 1$ which we have excluded, so we can assume that $s \neq s'$. Then the expressions $\mathbf{t}$ and $\mathbf{t}'$ are distinct minimal expressions for w in the dihedral group generated by $\{s, s'\}$. The order of ss' must be a finite number m, because if it were infinite, we would be working in the infinite dihedral group, in which all elements have unique minimal expressions, see example II.4. Hence $\mathbf{t}$ and $\mathbf{t}'$ of length $d = m$, have the form

$$(t_1, ..., t_d) = \begin{cases} (s, s', ..., s, s') & \text{if } m \text{ is even, or} \\ (s, s', ..., s, s', s) & \text{if } m \text{ is odd.} \end{cases}$$

$$(t'_1, ..., t'_d) = \begin{cases} (s', s, ..., s', s) & \text{if } m \text{ is even, or} \\ (s', s, ..., s', s, s') & \text{if } m \text{ is odd.} \end{cases}$$

In other words $F_w((t_1, ..., t_d)) = a(s, s')$ and $F_w((t'_1, ..., t'_d)) = a(s', s)$, and hence by hypothesis $F_w(\mathbf{t}) = F_w(\mathbf{t}')$ as required.

[6, chapter IV, section 1, proposition 5] ■

Theorem II.4. *If a group W generated by S satisfies **(E)** then it satisfies **(C)**.*

Proof. Let G be a group, and $f : S \mapsto G$ a function such that $(f(s)f(s'))^m = \varepsilon$ whenever ss' is of finite order m in W. Such a map satisfies the hypotheses of proposition II.4, so there exists a map g extending f to W such that $g(w) = f(t_1) \cdots f(t_d)$ whenever $(t_1, ..., t_d)$ is a minimal expression for w. To show that W admits a presentation of the required form, that is, it satisfies **(C)**, it is sufficient to show that g is a homomorphism because the claim then follows from the first isomorphism theorem. That g is a homomorphism follows from

$$g(sw) = f(s)g(w)$$

for all $s \in S$ and $w \in W$, since S generates W. This formula follows from proposition A, since this says that there are only two possible cases:

1) $l(sw) = l(w)+1$, in which case, if $(t_1, ..., t_d)$ is a minimal expression for w, then $(s, t_1, ..., t_d)$ is a minimal expression for sw, and the formula follows from the definition of g.
2) $l(sw) = l(w) - 1$, in which case set $w' = sw$, then $w = sw'$ and $l(sw') = l(w') + 1$ and by the first case $g(w) = f(s)g(sw)$, and hence $f(s)g(w) = g(sw)$ since $(f(s))^2 = \varepsilon$.

[6, chapter IV, section 1, theorem 1] ■

The key to this proof is showing that the function g is well-defined, i.e. that it does not depend on the choice of minimal expression for w. The proposition is merely bookwork to establish that the function satisfies certain conditions, there real power of the proof is in the lemma which establishes that the function is constant on the set of minimal expressions for a given element w.
This completes the proof of proposition C.

3D Proof of Theorem II.1

We have just proved that **(C)** and **(D)** are equivalent. To justify that Coxeter groups describe all possible combinatorial groups we need to prove that **(C)** and **(A)** are equivalent. That **(C)** *implies* **(A)** requires only that one check that the actions ρ defined by (II.2) satisfy the relations of a Coxeter system. This is a simple exercise which we leave to the reader. As a hint, K. Brown proves it in the case of a particular group in [8, capter II, section 2C], and the general case is essentially the same.

To complete the proof, all we need to do is show that **(A)** *implies* **(D)**. This can be proved independently of the material in chapter I [8, chapter II, section 1, corollary], but for brevity we shall make use of our work there. By the way that we derived condition **(A)** it should not be a surprise that the material in the first chapter applies to groups which satisfy **(A)**. We need to define the notion of a chamber with respect to the abstract set of mirrors $\mathcal{H} \times \{\pm 1\}$. One cannot use the obvious direct analogue of definition I.5, since there is no simple way to incorporate the non-emptiness condition, and so you would end up with too many cells, most of which were "empty". Instead we proceed as follows.

Definition II.12. Let W be a combinatorial group with generators S which satisfies **(A)**. Let the subset of $\mathcal{H}$ which corresponds to S be $\{H_1, ..., H_n\}$ then the **fundamental chamber** is the n-tuple $C = (H_1^+, ..., H_n^+)$, where H_i^+ abbreviates the half-space $(H_i, +1)$ in $\mathcal{H} \times \{\pm 1\}$. With the obvious generalisation of notation, the n-tuple $(H_1'^\sigma, ..., H_n'^\sigma)$[5], with $H_i'^\sigma = H_i'^+$ or $H_i'^-$ and $H' \in \mathcal{H}$, is a **chamber** if there is $w \in W$ such that $(H_1'^\sigma, ..., H_n'^\sigma) = \rho_w(C) := (\rho_w(H_1^+), ..., \rho_w(H_n^+))$.

The definitions of wall, adjacent, gallery, *et cetera* now carry straight over from before. The proofs of proposition I.1 and theorem I.1 require only the notions of chambers, adjacency, and galleries with the exception of step 1 of the proof of the theorem:

Step 1:

> Let D be a chamber with wall H, and let s be reflection with respect to H. Then $\rho_s(D)$ and D are adjacent along H, and moreover they are distinct.

Indeed, let $D = (H_1'^\sigma, ..., H_n'^\sigma)$, with $H = H_1'$ the mirror corresponding to $\rho_w(H_1^+)$. Then

$$\rho_s(D) = (\rho_s(H_1'^\sigma), ..., \rho_s(H_n'^\sigma)) \overset{\text{(II.2)}}{=} ((H_1'^{-\sigma}), ..., (sH_n'^\sigma))$$

so indeed D and $\rho_s(D)$ are adjacent. They are also clearly distinct.

This means that those results hold, and in particular, the proof of step 5 shows that **(D)** is satisfied. This completes the proof of theorem II.1.

Notes

1) This chapter does not follow any one source, it amalgamates material from [4], [6], [8], and [9] in the main. The exposition throughout is our own.
2) The overall scheme we use to prove theorem II.1 follows that in [8], however we have diverged from his approach in many of the details. The pleasing interplay between propositions A, B, C, and the proof of this theorem arose naturally during the course of writing this chapter, and was not planned.
3) The proof of proposition B requires lemma II.5, but instead we prove its cousin II.4. We came upon lemma II.4 while searching for a proof of lemma II.5 which avoided root systems. While that search was ultimately unsuccessful, we chose to include the related result and its somewhat long and involved proof because it allowed us to include the strong exchange condition which was useful later. We have not seen the connection between these two lemmata mentioned anywhere.

[5] Note that this tuple will contain elements from the whole of $\mathcal{H} \times \{\pm 1\}$, and not just the half-spaces of $\{H_1, ..., H_n\}$. We are defining a chamber by its "walls", and the associated half-spaces, noting that the rest of the "hyperplanes" in $\mathcal{H}$ are not needed to define that chamber.

4) We have not seen the approach given in the last section anywhere else. It is quite obvious why, but this notational minefield will illustrate the usefulness of the reflection representation in the next chapter.

Chapter III

Coxeter Groups: A Good Class of Combinatorial Groups

In the previous chapter we introduced and discussed combinatorial group theory in general, and in particular we highlighted many fundamental problems with approaching group theory in this way. We then went on to introduce Coxeter groups as combinatorial reflection groups, and spent quite a bit of time proving some combinatorial results about Coxeter systems. In this chapter we shall introduce a construction called the reflection representation which will go quite some way to clearing up the concerns we had for combinatorial groups in the particular case of Coxeter groups. The reflection representation is the natural representation of a reflection group on a real vector space. We shall prove Tits' Theorem which in one go shows that the representation has all of the properties we could ask of it. With this we shall be able to bring the geometry of the first chapter into the ideas of the second chapter, and see that our proofs can be significantly streamlined. Indeed combinatorial questions which seemed forbidding, if not almost impossible, suddenly become almost trivial. We shall see this in the case of the word problem, where we shall be able to give two geometric solutions. We shall also be able to study and classify the class of finite Coxeter groups, which is of fundamental importance to the study of Coxeter groups in general.

III.1 The Reflection Representation

1A The Definition

In section 2B of the previous chapter we introduced the Coxeter matrix, promising that it would serve us as more than just a typographical convenience (indeed one would hope so, since it was immediately superseded in that regard by the Coxeter diagram). Now that time has come with the following definition.

Definition III.1. Given a Coxeter matrix M with entries m_{ij} of rank n, let V be a real vector space of dimension n with basis $\{e_1, \ldots, e_n\}$. Define on V the **symmetric bilinear form** B via

$$B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right)$$

That this is a sensible definition to make can be seen by comparing this to equation (I.1) in theorem I.2.

We shall use this bilinear form to make W, the Coxeter group associated to M, act on V, such that the generators of W, S (which are determined by M) act as linear reflections with respect to B. To be rigorous, one must worry about the possible degeneracy of B, but this turns out not to be a problem, see [6, chapter V, section 4, proposition 1]. Recall the equation for a reflection in Euclidean space given in definition I.2. Let $1 \leq i \leq n$, and define

$$\sigma_i : V \mapsto V : v \mapsto v - 2B(v, e_i)e_i$$

Each σ_i is a linear reflection of V in the hyperplane orthogonal[1] to e_i, so $\sigma_i \in GL(V)$, where $GL(V)$ is the group of matrices which act on V as linear transformations (so if $V = \mathbb{R}^n$, $GL(V) = GL_n(\mathbb{R})$).

Definition III.2. Let (W, S) be a Coxeter system, let V be a real vector space of dimension $\#S$, with corresponding symmetric bilinear form B. The **reflection representation** of (W, S) is given by the homomorphism $\rho : W \mapsto GL(V)$ which maps the generators S by

$$s_i \mapsto \sigma_i.$$

Remark III.1. For an arbitrary group G, a representation is a vector space V along with a homomorphism $\rho : G \mapsto GL(V)$. A representation can also be thought of as a way of getting a group to act on a vector space by geometric transformations, which is why we are using one here. Another interpretation is that we are realising the group as a subgroup of a group of matrices in some sense. We shall not need to consider the reflection representation in the broader context of representation theory, but the interested reader can find an excellent introduction to the subject in [24]. We record some of the basic definitions from representation theory below as they will be relevant to us.

Definition III.3. Let G be a group, V a vector space, and ρ a representation of G on V. ρ is **faithful** if it is injective. Since ρ is necessarily surjective onto its image, and the image of a group under a homomorphism is a group, this means that G is *isomorphic* to a subgroup of $GL(V)$. If G has a faithful representation, it is called a **linear group**.

ρ is **irreducible** if it leaves no non-trivial subspace of V invariant. In other words, there is no proper subspace of V, say $\{0\} \subsetneq V' \subsetneq V$, such that for all $v \in V'$, and for all $g \in G$, the action of the matrix $\rho(g) \in GL(V)$ takes v to another vector in V'. If such a an invariant subspace *does* exist, we say ρ is **reducible**. Then ρ is **completely reducible** if V decomposes into a direct sum of subspaces, such that ρ restricted to each is an irreducible representation.

Lemma III.1 (Macshke's Theorem)**.** *If ρ is a reducible representation of a finite group, then it is completely reducible. [24, lemma 3.13]* ■

This is a standard result from representation theory, the proof of which we omit.

Remark III.2. We have now seen three definitions which are, or at least seem to be, related. A collection of hyperplanes being *essential* (definitions I.4 and I.12), a Coxeter system being *irreducible* (definition II.9), and a representation being *irreducible* (definition III.3).

It is clear that the reflection representation is irreducible if and only if the collection of hyperplanes in V associated to the reflections σ_i is essential. On the other hand there is no logical connection between a Coxeter system being irreducible, and its reflection representation being irreducible (one can construct examples which are one but not the other, and *vice versa*)[2]. This lexical annoyance is something which must be born in mind throughout our discussion, particularly in theorem III.4 and its proof.

We can use the reflection representation to prove two results which naively are so obvious that they are not worth mentioning, and indeed which we have tacitly assumed to be true so far.

[1] Orthogonal with respect to B.

[2] It turns out that a Coxeter group is irreducible if and only if a representation U defined in [6, p. 87] is irreducible, so this clash of terminology has arisen for justifiable reasons.

Proposition III.1. *Let (W, S) be a Coxeter system with presentation $\langle s_1, ..., s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = \varepsilon\rangle$ as usual. Then*

1) *each s_i represents a distinct non-trivial element of W*[3]*, and*
2) *the order of $s_i s_j$ is m_{ij}*[4]*.*

Proof. Clearly σ_i and σ_j act as different reflections of V, since e_i and e_j are different basis elements for $i \neq j$. The homomorphism ρ guarantees that s_i and s_j are therefore distinct and non-trivial, proving (1).

Let $i \neq j$ be chosen such that m_{ij} is finite, and write $V_0 = \mathbb{R}e_i \oplus \mathbb{R}e_j$. Let V_1 be the orthogonal complement of V_0 in V. Both V_0 and V_1 are invariant under the action of σ_i and σ_j, which generate the dihedral group $D_{m_{ij}}$ acting on V_0, and leaving V_1 fixed. This means that $\sigma_i\sigma_j$ has order m_{ij}, and so by ρ, the order of $s_i s_j$ must be a multiple of m_{ij}; but we already know that it is a divisor, so we must have that the order of $s_i s_j$ is exactly m_{ij}. What if m_{ij} is infinite. With the same decomposition of V as above, σ_i and σ_j still leave V_0 invariant, so consider the action restricted to V_0. One may then explicitly calculate $\sigma_i\sigma_j(v)$, for some general $v = \alpha e_i + \beta e_j$, noting that $B(e_i, e_j) = -1$ and $B(e_i, e_i) = B(e_j, e_j) = 0$ and thereby show that it has infinite order. [8, section II.5, theorem A] ∎

1B Tits' Theorem

We can construct the reflection representation for any combinatorially defined Coxeter system, and it shows that every Coxeter group is homomorphic to a geometric group generated by reflections. We would ideally like to use this to apply the ideas of chapter I to Coxeter groups. In order to be able to do this, we need the answers to the following two questions to be yes:

1) Is ρ an *isomorphism* onto its image?
2) Does W act discretely on V via ρ?

More remarkable than the fact that the answer to both is yes for *every* Coxeter system, is that both results follow almost immediately from the same theorem. Before we can state this, we need another definition from representation theory.

Definition III.4. Let G be a group, V a vector space with dual space V^*, and let $\rho : G \mapsto GL(V)$ be a representation of G. Then the **dual representation** of ρ is $\rho^* : G \mapsto GL(V^*)$, defined by $(\rho^*(g)f)(v) = f(\rho(g^{-1})v)$ for $v \in V$, and $f \in V^*$. g^{-1} is needed instead of simply g to make sure ρ^* is a well-defined homomorphism.

For our purposes V is always finite dimensional. We have a basis of V, so we can identify V^* with V when B is non-degenerate. One might wonder therefore why we bother considering the dual[5] representation of the reflection representation and not just work with the reflection representation directly. The answer is that there is no easy way to define a fundamental domain for the action of W on V in general since there is no natural orientation on the hyperplanes corresponding to each σ_i. In V^* on the other hand we can make the following definitions:

Definition III.5. For each $s \in S$ write

$$H_s = \{f \in V^* \mid f(e_s) = 0\}$$

[3](2) of example II.2 shows that two ostensibly different generators of a combinatorial group may turn out to be the same element.

[4]The corresponding relation in the presentation only guarantees that the order of $s_i s_j$ *divides* m_{ij}.

[5]In some books, the dual of the reflection representation is called the **contragredient representation**.

$$A_s = \{f \in V^* \mid f(e_s) > 0\}$$

which is the hyperplane in V^* (respectively the positive half-space in V^*) corresponding to s. Then the **fundamental chamber** of ρ^* is $C = \bigcap_{s\in S} A_s$. C is an open simplicial cone in V^* (see lemma I.1[6]). The translates of C under $\rho(W)$ are called the **chambers** of ρ^*.

We shall abbreviate $\rho^*(w)C$ to wC henceforth to simplify notation. It is unambiguous both that, and how w is acting on C.

Definition III.6. Let (W, S) be a Coxeter system, with dual reflection representation ρ^* on V^*. Let C be the fundamental chamber for this action, then the **Tits cone** is the subset of V^* given by

$$U = U(W) = \bigcup_{w\in W} w\overline{C}$$

Theorem III.1 (Tits' theorem)**.** *If (W, S) is a Coxeter system, with the notation above, $C \cap wC = \emptyset$ whenever $w \neq \varepsilon$.*

This is the equivalent result to step 5 of the proof of theorem I.1, however we cannot just use that proof because we cannot assume the discreteness condition is satisfied if W is infinite. For the proof we follow [16, chapter I, section 2]. We shall reduce to the case that W is a dihedral group, in which case the following lemma will come in useful.

Lemma III.2. *Let $T = \{s, s'\}$ be a subset of S so that $W_T \subseteq W$ is a dihedral group (see 3 of example II.4), and suppose $w \in W_T$. Then either*

a. *$w(A_s \cap A_{s'}) \subseteq A_s$ and $l_T(sw) = l_T(w) + 1$, or*
b. *$w(A_s \cap A_{s'}) \subseteq sA_s$ and $l_T(sw) = l_T(w) - 1$.*

Proof. For ease of notation, we can assume $W = W_T$ and $V = \mathbb{R}e_s \oplus \mathbb{R}e_{s'}$. Writing m for the order of ss' in W, we consider two cases:

Case i: m is finite, so V is Euclidean and we can identify V with V^*. The result then follows from 3 of proposition I.1.

Case ii: m is infinite. Let $\{f, f'\}$ be the basis of V^* dual to $\{e_s, e_{s'}\}$. We can then explicitly compute the action of $\{s, s'\}$ on $\{f, f'\}$ via ρ^*. We find

$$\rho^*(s)f = -f + 2f', \quad \rho^*(s)f' = f', \quad \rho^*(s')f = f, \quad \rho^*(s')f' = 2f - f'.$$

Let L be the affine line $\{tf + (1-t)f' \mid t \in \mathbb{R}\}$ which connects f and f'. The equations above show that L is stable under the action of $\rho^*(W)$, so s and s' act on L by reflections in f and f' respectively (see example I.5). Writing I for the open line segment connecting f and f', it is clear that $A_s \cap A_{s'} = \bigcup_{\lambda>0} \lambda I$ from which the claim follows. [16, chapter 1, lemma 2.2] ■

The general case of this lemma, from which the theorem will follow, is as follows.

Lemma III.3. *Let $w \in W$ and $s \in S$, then either*

1) *$wC \subseteq A_s$ and $l_T(sw) = l_T(w) + 1$, or*
2) *$wC \subseteq sA_s$ and $l_T(sw) = l_T(w) - 1$.*

Proof. We shall do induction on $n = l(w)$. Consider the statements (P_n):

If $w \in W$ with $l(w) = n$, and $s \in S$, then either

[6] Since the e_i's form a basis of V, it follows easily that $\{H_s\}_{s\in S}$ is an essential collection of hyperplanes in V^*.

i. $wC \subseteq A_s$, or

ii. $wC \subseteq sA_s$ and $l(sw) = l(w) - 1$.

and (Q_n):

> For any $s \neq s' \in S$, let $T = \{s, s'\}$, and suppose $w \in W$ with $l(w) = n$, then there is $u \in W_T$ such that $wC \subseteq u(A_s \cap A_{s'})$ and $l(w) = l(u) + l(u^{-1}w)$.

If $n = 0$, $w = \varepsilon$, and taking $u = \varepsilon$ we see that both (P_0) and (Q_0) hold. We do the induction in two steps:

Step 1: (P_n) and (Q_n) implies (P_{n+1}).

If $l(w) = n + 1$, we can write $w = s'w'$ such that $l(w') = n$ for some $s' \in S$. If $s = s'$ then by (P_n), $w'C \subseteq A_s$ so $wC \subseteq sA_s$ and $l(sw) = l(w') = l(w) - 1$ which is case (ii) in (P_{n+1}).

If $s \neq s'$ we can apply (Q_n) to w', so there exists $u \in W_T$ satisfying $w'C \subseteq u(A_s \cap A_{s'})$ such that $l(w') = l(u) + l(u^{-1}w')$. Hence

$$wC \subseteq s'u(A_s \cap A_{s'})$$

By lemma III.2 we have either

a. $s'u(A_s \cap A_{s'}) \subseteq A_s$ and $l(ss'u) = l(s'u) + 1$, or

b. $s'u(A_s \cap A_{s'}) \subseteq sA_s$ and $l(ss'u) = l(s'u) - 1$.

If (a) is the case, $wC \subseteq A_s$ and we get case (i) of (P_{n+1}). If (b) is the case we have $wC \subseteq sA_s$, and we must check that $l(sw) \leq l(w)$, from whence case (ii) of (P_{n+1}) will follow. We have

$$\begin{aligned} l(sw) &= l(ss'w') = l(ss'uu^{-1}w') \\ &\leq l(ss'u) + l(u^{-1}) \\ &= l(s'u) - 1 + l(w') - l(u) \\ &\leq l(w') = l(w) - 1 \end{aligned}$$

Step 2: (P_{n+1}) and (Q_n) implies (Q_{n+1}).

Let $w \in W$ with $l(w) = n + 1$. If $wC \subseteq (A_s \cap A_{s'})$ then take $u = \varepsilon$ and we are done. Otherwise we can use (P_{n+1}) to say that $wC \subseteq sA_s$ with $l(sw) = l(w) - 1$; writing $w' = sw$, $l(w') = n$. Applying (Q_n), we know there is $u \in W_T$ such that $l(sw) = l(u) + l(u^{-1}sw)$ and $swC \subseteq u(A_s \cap A_{s'})$. Hence $wC \subseteq su(A_s \cap A_{s'})$. To complete the step we must check that $l(w) = l(su) + l(u^{-1}sw)$.

$$l(w) = l(suu^{-1}sw) \leq l(su) + l(u^{-1}sw)$$

but on the other hand

$$l(w) = l(sw) + 1 = l(u) + l(u^{-1}sw) + 1 \geq l(su) + l(u^{-1}sw)$$

This completes the induction. That (P_n) holds for all n almost gives the statement of the lemma; we need only check that in case (i) we also have that $l(sw) = l(w) + 1$. In this case, we have $swC \subseteq sA_s$, so by case (ii) $l(w) = l(ssw) = l(sw) - 1$ and hence $l(sw) = l(w) + 1$ as required. [16, chapter 1, lemma 2.3] ■

Proof of Tits' theorem. If $w \neq \varepsilon$ then we can write $w = sw'$ for some $s \in S$ such that $l(w) = l(w') + 1 = n + 1$. By the principle of the excluded middle, lemma III.3 says that $w'C \subseteq A_s$, and hence $wC = sw'C \subseteq sA_s$. Since $C \subseteq A_s$ we have that $C \cap wC = \emptyset$ as required. [16, chapter 1, theorem 2.1] ■

1C Consequences of Tits' Theorem

By the way chambers of ρ^* were defined, the following result is immediate.

Corollary III.1. *W acts simply-transitively on the set of chambers of ρ^* (compare to statement (1) of theorem I.1, this means the action of W is simply-transitive). [16, chapter 1, corollary 2.4]* ■

Theorem III.2. *The reflection representation and its dual are faithful for any Coxeter system (W, S). This means that Coxeter groups are linear, and the answer to the first question is yes.*

Proof. If $\rho^*(w) = \varepsilon$, then $wC = \rho^*(w)C = C$, and hence $w = \varepsilon$ by Tits' theorem. The kernel of ρ^* is therefore trivial, and so it is injective. From the definition of ρ^* in terms of ρ it is clear that the kernel of ρ is contained in the kernel of ρ^*, and hence ρ is also injective. [16, chapter 1, corollary 2.5] ■

Theorem III.3. *ρ^* acts discretely on the interior of the Tits cone U, the corresponding action of ρ is also discrete. This means that the answer to the second question is yes.*

Proof. By transport of structure, it is sufficient to show that ρ^* is a discrete action, in particular that the induced topology on $\rho^*(W)$ in $GL(V^*)$ is the discrete topology. Let $w \in W$, and $f \in C$, and define $Y = \{g \in GL(V^*) \mid g(f) \in C\}$, an open neighbour of the identity in $GL(V^*)$. By Tits' theorem $\rho^*(W) \cap Y = \{id\}$, and so $\rho^*(W)$ has the discrete topology in $GL(V^*)$. [6, chapter V, section 4, no. 4, corollary 3] ■

These results mean that the reflection representation forms a bridge between chapters I and II. We can apply this to solve the Word Problem in Coxeter groups, as we shall see in the next section. However we need to reconcile the picture we built up for infinite reflection groups in the first chapter with the picture we have here. In that chapter we said that an infinite reflection group needed to consist of reflections in affine hyperplanes in order to be discrete, however here we have possibly infinite groups generated by reflections in hyperplanes which all pass through the origin, and nevertheless we claimed in the last theorem that the action was discrete.

The key is that we were very careful to say that the the action in the case of ρ^* was discrete *on the interior of the Tits cone*, in particular there is no contradiction if U is not the whole of V^* and its interior does not contain the origin. We shall show that this is in fact the case[7].

Lemma III.4. *Suppose $x \in -C$ (where $-C$ is the image of C in V^* under the antipodal map). Then $x \in U$ if and only if W is finite.*

Proof. Suppose $x \in U$, so $x = \rho^*(w)y$, for some $y \in C$. By lemma III.3, $l(ws) < l(w)$ if and only if C and wC lie on opposite sides of H_s, so since C and $-C$ lie on opposite sides of H_s for all $s \in S$, w is the unique longest element in W, and so W is finite by proposition B and its proof. Conversely, if W is finite, take w its longest element, then $wC = -C$, and so $x \in U$. [13, lemma D.2.3] ■

Corollary III.2. *If W is infinite, U contains no line through 0. Moreover, the following are equivalent:*

1) *W is finite,*
2) *U is the whole of V^*, and*
3) *0 is in the interior of U.*

[7] In [1, section 2.7], P. Abramenko and K. Brown axiomatise the infinite hyperplane arrangements which admit the structure and theory which we developed in chapter I.

Proof. The first claim is obvious, since we could choose any chamber of U to be the fundamental chamber. The only implication which is not immediate is that (3) *implies* (1). By the lemma, if W is infinite, there is $x \in V^*\backslash\{0\}$ which is not in U, it follows that λx is not in U for all $\lambda > 0$. Taking the limit as λ goes to 0 we see that 0 is not in the interior of U. [1, proposition 2.91], [16, chapter 1, proposition 2.7], and [13, corollary D.2.4] ■

We can in fact write down the interior of U explicitly. Recall that the walls of C are hyperplanes which correspond bijectively with the generators S, we shall denote the hyperplane corresponding to $s \in S$ by C_s[8]. A facet of C is characterised by the maximal set of hyperplanes such it is in their intersection, so we have one facet for each subset $T \subseteq S$, we shall denote that facet C_T. Note that these facets are open sets.

Definition III.7. Denote by $\mathcal{S}$, the collection of subsets $T \subseteq S$ such that T generates a finite special subgroup W_T of W. Then we set

$$C^\circ = \bigcup_{T \in \mathcal{S}} C_T$$

and put

$$\mathcal{I} = \bigcup_{w \in W} wC^\circ \subseteq U. \tag{III.1}$$

Lemma III.5. *$\mathcal{I}$ is the interior of U.*

Proof. $\mathcal{I}$ is a union of open sets, so is open in U, and hence is in the interior of U. Conversely suppose that $x \in \overline{C}\backslash C^\circ$. Let T be the subset of S corresponding to the hyperplanes C_s which contain x, then we know that W_T is infinite. Applying the equivalence of (1) and (3) to the Tits cone $U(W_T)$ associated to the representation $\rho^*(W_T)$, we see that the "cone-point" of $U(W_T)$ is not in its interior, so x is not in the interior of U, since it is the image of this cone-point under the natural inclusion of $U(W_T)$ in U. [13, theorem D.2.6(iii)] ■

How do we use this to recover the picture we had in chapter I? If W is finite, then W acts "properly" on the whole of V^* and the chambers are simplicial cones. If W is infinite, it acts "properly" in the interior $\mathcal{I}$ of its Tits cone U, which is contained in an open half-space with respect to some hyperplane through the origin. *If* there were some W–invariant hyper-surface in V^* which was defined as the graph of a function on this hyperplane (or some similar definition), then we could intersect U with this hyper-surface; then W acts by reflections in "lines" (the intersections of the hyperplanes wC_s with the surface).

That is quite a big *if*, and a very vague sketch. We shall say something much more concrete in two special cases when we get to section 3B (we shall have to wait until we have considered the properties of the symmetric bilinear form B more closely). We have already seen that by assuming that W acted on the whole of V, we possibly excluded some infinite reflection groups. In fact, the class of infinite reflection groups which we discussed is called the class of *affine reflection groups* (or affine Weyl groups) for obvious reasons. There are other classes of infinite reflection group which do not fall into this category, for example hyperbolic reflection groups (discrete symmetries of hyperbolic space generated by reflections in hyperbolic lines)[9].

In anticipation however, we can still make use of what we have done in constructing U and proving the W acts on the chambers in U as we would hope. We can use the walls and chambers in V^* just as we discussed in the first chapter, and indeed the results we proved there will still be applicable. We shall first use this to discuss the Word Problem for Coxeter groups.

[8] Previously we were calling these H_s.

[9] These three together summarise the reflection groups in spaces of constant curvature: finite groups act on the sphere, affine groups on the plane, and hyperbolic groups on hyperbolic space. The symmetries of these geometries are covered extensively in [28].

III.2 The Word Problem

What makes the Word Problem so difficult? We introduced the length function for Coxeter systems, but this definition extends to any combinatorial group. In the case of dihedral groups (example II.4) we were able to take any word and put it in a standard form which was a minimal expression for that element, and which was (almost always) unique. Could we generalise this approach to other combinatorial groups? We may well be able to play about with the relations of the group and reduce a word to a minimal expression, and know that we have done so, but we shall not be so lucky that such minimal expression will be unique. In order to solve the Word Problem one would need to have a procedure by which, given a minimal expression, one could use the relations to write down all of the other minimal expressions for that element. Then, given another word, one could find a minimal expression for it and check whether it is on the list. The big problem comes in writing down that list. There is no reason *a priori* why one should necessarily be able to transform one minimal expression into another minimal expression without lengthening the word at some intermediate step. If the group is large or even infinite, the length increases required could be arbitrarily large, and so one would have no way of checking that all minimal expressions have indeed been found. One might compare it to potential wells in a physics problem. Minimal expressions are the local minima, but to get from one local minima to another could take a very large amount of activation energy.

For Coxeter groups, the Word Problem is soluble; and this is because it turns out that one may transform any minimal expression for an element into any other without increasing the length of the word in the interim, and moreover, there are finitely many minimal expressions for each element. This is achieved by a series of so-called M-operations, and was done by Tits in [27]. It is discussed in detail in, for example [8, pp. 49–51] or [9, pp. 60–62]. The proof is completely combinatorial, uses the condition **(E)**, or the equivalent condition **(D)**, and gives an explicit algorithm, based on the one outlined in the preceding paragraph, to check whether two words correspond to the same element. Here we shall discuss two much more quickly and easily stated solutions to the Word Problem, both of which are geometric, but which as a result, do not reveal the structure of the group so plainly as does Tits' solution.

The first solution of the Word Problem we shall look at is based on theorem III.2 [9, section 4.3, paragraph 1]. Fix a Coxeter system (W, S), and compute the matrices $\rho(s_i)$ for all $s_i \in S$. Now take two words in the alphabet S which we want to compare: $(t_1, ..., t_k)$ and $(t'_1, ..., t'_{k'})$, with $t_i, t'_j \in S$ for all i and j, and compute the matrix products $\rho(t_1) \cdots \rho(t_k)$ and $\rho(t'_1) \cdots \rho(t'_{k'})$. Since W is isomorphic to its image in $GL(V)$, the group elements $w = t_1 \cdots t_k$ and $w' = t'_1 \cdots t'_{k'}$ are the same if and only if these two matrices are the same. This method in fact shows that the Word Problem is soluble for any linear group.

What makes Coxeter groups special amongst linear groups in this regard is that with theorem III.2 behind us, we have the full power of the discussion in chapter I, since any Coxeter group is isomorphic to a geometric reflection group. Thus we can use the exegesis first introduced in remark I.5, which also holds for infinite reflection groups. Let (W, S) be a Coxeter system, and construct its reflection representation ρ on a vector space V. W is isomorphic to its image $\rho(W)$ which is a geometric reflection group. Let X be the poset of cells arising from this geometric reflection group, and choose a fundamental chamber C. If we label this with the identity, then we can label each chamber D in X by the unique $w \in W$ which takes C to that chamber, i.e. the w such that $D = wC$. Then galleries from C to wC correspond to words in S which are expressions for w, via step 3 in the proof of theorem I.1. We can now reinterpret proposition I.1 as talking about minimal words. The length function for words now coincides with the combinatorial distance function on the chambers (see definition I.10) via

$$l_S(w) = d(C, wC)$$

and in fact, this suggests that we can define a distance function on the combinatorial group W via [8, p. 34]

$$d(w, w') := d(wC, w'C) \equiv l(w^{-1}w').$$

How can we use this to solve the Word Problem? Given two words $(t_1, ..., t_k)$ and $(t'_1, ..., t'_{k'})$, they define galleries

$$\Gamma : C, t_1C, t_1t_2C, ..., t_1t_2 \cdots t_kC \text{ and } \Gamma' : C, t'_1C, t'_1t'_2C, ..., t'_1t'_2 \cdots t'_{k'}C$$

which define two paths through X starting at C. If the two paths end at the same chamber, the two words correspond to the same group element, and if they end at different chambers, they correspond to different elements. We shall return to this way of thinking about things in the next section.

One may wish to think through this method in the case of D_4, comparing example I.2 and (3) of example II.4.

III.3 Classification of Finite Coxeter Groups

The classification of finite Coxeter groups was first done by H. S. M. Coxeter himself [12]. The proof consists of repeated applications of the criterion for a Coxeter system to be finite below. Theorem II.2 tells us that we only need to consider irreducible Coxeter systems, because a reducible Coxeter system is clearly finite if and only if all of the direct summands in its decomposition into irreducible components are finite. We include this classification in a chapter aimed at justifying that Coxeter groups are 'a good class of combinatorial groups' for a number of reasons. The first is that the proof is very closely related to the reflection representation, the main tool of this chapter; the second is that it is unusual that such a classification would exist give an arbitrary class of combinatorial groups; and the third reason is that the classification theorem is very useful in the study of Coxeter groups in general, and indeed we shall see an application of it at the start of the next section.

3A Witt's Criterion

Theorem III.4 (Witt's criterion[10]). *Let (W, S) be an irreducible Coxeter system, V a real vector space of dimension $\#S$, with related bilinear form B. Then W is finite if and only if B is positive-definite.*

For the proof we shall need a definition and some lemmata.

Definition III.8. Let V be a vector space with symmetric bilinear form B. The **radical** of B is the subspace of V orthogonal to V:

$$V^{\perp} = \{v \in V \mid B(v, u) = 0 \text{ for all } u \in V\}$$

If B is degenerate, then the eigenspace of the matrix of B corresponding to the eigenvalue 0 will indeed satisfy the condition of the definition. The zero vector is always in the radical.

Lemma III.6. *Let W be an irreducible Coxeter system with associated bilinear form B. If V' is a proper subspace of V which is invariant under the action of the reflection representation of W, ρ, then V' is contained in $V^{\perp}$.*

[10]Proved by E. Witt in "Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe" (1941).

Proof. Let $x \in V'$, then for all $i \in \{1, ..., n\} = I$ where $n = \dim V$, $2B(x, e_i)e_i = x - \rho(s_i)x \in V'$. We want to show that $x \in V^\perp$, so it is sufficient to show that $e_i \notin V'$, because this will force $B(x, e_i) = 0$.

Let $J = \{i \in I \mid e_i \in V'\}$; we want to show that $J = \emptyset$. Since V' is a proper subspace of V, $J \neq I$. Assume $J \neq \emptyset$, then by the definition of W being irreducible, there is $t \in I \backslash J$ and $u \in J$ such that $2B(e_u, e_t) \neq 0$. Hence $2B(e_u, e_t)e_t = e_u - \rho(s_t)e_u \in V'$. Therefore e_t itself is in V', and $t \in J$, a contradiction. So we must have that $J = \emptyset$ as required. [9, proposition 2.3.7] ■

Lemma III.7. *Let (W, S) be irreducible, and let ρ be its reflection representation; then*

1) *if B is degenerate, ρ is* not *completely reducible*[11]*, or*
2) *if B is non-degenerate, ρ is irreducible.*

Proof.

1) It is a trivial exercise to show that ρ leaves B invariant, i.e. that $B(\rho(w)v, \rho(w)v') = B(v, v')$ for all $w \in W$, and hence $V^\perp$ is an invariant subspace. Since $B(e_s, e_s) = 1$ but B is degenerate, $V^\perp$ is proper and non-trivial. Suppose ρ were completely reducible, then there is another invariant subspace of V, V', such that $V = V^\perp \oplus V'$, but by lemma III.6 $V' \subseteq V^\perp$, a contradiction. Hence ρ is not completely reducible.
2) If B is non-degenerate, $V^\perp = \{0\}$, and lemma III.6 guarantees that any proper invariant subspace vanishes.

[16, chapter 1, corollary 1.17] ■

Finally we need a lemma from linear algebra which we give without proof.

Lemma III.8. *Let G be a group and ρ an irreducible representation of G on a vector space V. Assume that there is an element $g \in G$ such that $\rho(g)$ acts on V by a reflection.*

1) *Every linear map $V \mapsto V$ which commutes with $\rho(G)$ is a scalar multiple of the identity map.*
2) *If V is finite dimensional, and $\mathscr{B}$ is a non-zero, $\rho(G)$-invariant bilinear form, then $\mathscr{B}$ is either symmetric or anti-symmetric, and every $\rho(G)$-invariant bilinear form on V is proportional to $\mathscr{B}$.* ■

Proof of Witt's criterion. Assume W is finite, so Maschke's theorem (lemma III.1) says that if ρ is reducible, it is completely reducible. By the principle of the excluded middle, lemma III.7 guarantees that ρ is irreducible.

Let $\mathscr{B}$ be an arbitrary positive-definite symmetric bilinear form on V (such a form always exists), and let $\mathscr{B}'$ be the sum of its transforms under ρ, so $\mathscr{B}'$ is non-zero and $\rho(W)$–invariant. V is finite dimensional since W is finitely generated so we may apply lemma III.8 to conclude that $B = k\mathscr{B}'$, for some $k \in \mathbb{R}$. Since $\mathscr{B}$ is positive-definite, it follows that B is negative-definite, zero, or positive-definite. The observation that $B(e_s, e_s) = 1$ for any $s \in S$ guarantees that B is positive-definite, as required.

Now assume that B is positive-definite. Then the orthogonal subgroup $O(n, B) \subseteq GL(V)$ is compact. By the proof of theorem III.3, $\rho(W)$ is a discrete subgroup of $O(n, B)$, and hence finite. Theorem III.2 says that $\rho(W) \cong W$, and hence W itself is finite, completing the proof. (Adapted from [9, proposition 5.3.2], [16, chapter 1, theorem 4.1], and [6, chapther V, section 4, theorem 2]) ■

[11] See definition III.3.

Remark III.3. The literature does this result quite a disservice. Of the works cited in the bibliography, only three use this result to prove the classification theorem and attempt a proof[12]. The proofs in [9] and [16] are both wrong, while the proof in [6] is inscrutable in parts to anyone below postgraduate level. A. Cohen attempts to prove that, without the condition that W is finite, ρ being irreducible is sufficient to conclude that B is positive-definite, however refers back to a lemma (lemma 5.1.2(ii)) which requires that W is finite. It also requires that ρ is *absolutely irreducible*, the justification for which is buried in lemma 2.3.12 with no cross-reference. H. Hiller avoids this mistake, but at the same stage of the proof instead refers to a result (proposition A8) which assumes that B is positive-definite in order to prove that very fact. Between the three accounts we have managed to assemble a correct proof.

Theorem III.5 (Classification of Finite Coxeter Groups)**.** *Let (W, S) be an irreducible Coxeter system. Then W is finite if and only if the corresponding Coxeter diagram $\mathcal{V}$ is isomorphic to one of the diagrams in table III.1[13]. Moreover, no two of these groups are isomorphic.*

Proof outline. After proving Witt's criterion, the proof of the classification consists of a rather tedious checking of various cases, by deriving a contradiction from the fact that B is not positive-definite in each case. We shall only outline the various steps. Slightly more detail is given in [9, theorem 5.3.3], and yet more in [6, chapter VI, section 4].

Let (W, S) be an irreducible Coxeter system with n generators, and with Coxeter diagram $\mathcal{V}$.

Step 1: If $\mathcal{V}$ appears in table III.1, then W is finite.

For the other implication, we henceforth assume that W is finite.

Step 2: If $\mathcal{V}'$ is a subgraph of $\mathcal{V}$, then the Coxeter group corresponding to $\mathcal{V}'$ is finite. This means that we can exclude cases by excluding certain subgraphs; it follows from proposition II.1.

Step 3: $\mathcal{V}$ cannot contain a cycle.

Step 4: $\mathcal{V}$ cannot contain a vertex with valence greater than 3.

Step 5: If a vertex of $\mathcal{V}$ has valence 3, then all three incident edges have no label (label 3 omitted).

Step 6: If $\mathcal{V}$ has an edge with label at least 6, then $n = 2$.

Step 7: If $\mathcal{V}$ has a vertex with valence 3, then every edge has no label.

Step 8: $\mathcal{V}$ has at most one vertex with valence 3.

Step 9: If an edge of $\mathcal{V}$ has label 5, then edges on either side of it have no label.

Step 10: $\mathcal{V}$ does not contain any subgraph appearing in table III.2.

Step 11: The only graphs which satisfy steps 3–10 are listed in table III.1. ■

3B The Spaces on which Coxeter Groups Act

What can we say about the spaces on which a Coxeter group acts just by looking at its presentation? In the case of finite Coxeter groups we can say something quite strong now that we have the link between geometric and combinatorial reflection groups.

Theorem III.6. *An irreducible finite Coxeter group W acts essentially on a vector space V if and only if the dimension of V is equal to the number of reflections which generate W[14].*

[12] [17] and [13] give an alternative proof of the classification theorem, and [12] was published six years before Witt's paper.

[13] The group G_2 is listed separately from the other dihedral groups because it is also a so-called Weyl group.

[14] This result is weakened if we drop the irreducibility condition, but we can still say something sensible about the way W acts if we use some results about *rigidity*, for details, see the appendix A.3.

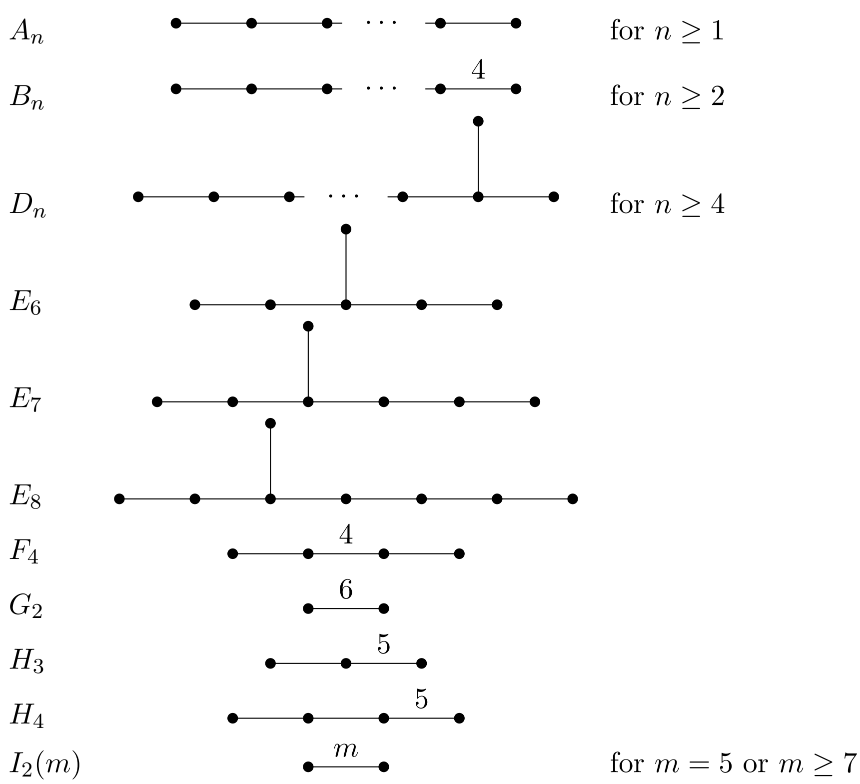


Table III.1: All Coxeter diagrams corresponding to finite Coxeter systems. The subscript number in the name of each indicates the number of vertices.

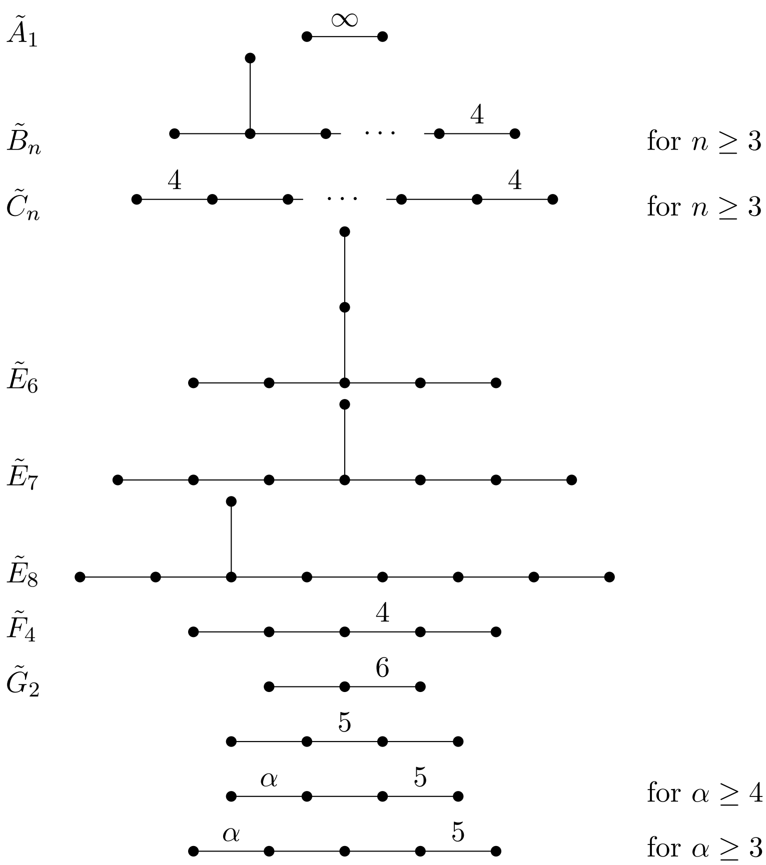


Table III.2: Some Coxeter diagrams corresponding to infinite Coxeter systems. The subscript n is 1 less than the number of vertices.

Proof. To prove the forward implication, assume that W acts essentially on V, then by theorem I.2 the associated chambers are simplicial cones. Lemma I.1 says then that each chamber has n walls, and since W is generated by reflections in the walls of any chamber, theorem I.1, W has a presentation with n generators (which is a Coxeter-type presentation by the proof that **(A)** *implies* **(C)**, in which the generators which appear in the statement of **(A)** are the same as those in the presentation eventually constructed for **(C)** — for details see the proofs of the various implications either given or cited throughout). By explicit calculation, one can see that no two non-isomorphic graphs which appear in table III.1 define isomorphic Coxeter systems, so it makes sense to say that the finite irreducible Coxeter *group* W has n generators.

For the other implication, assume V is an n–dimensional vector space and (W, S) is a finite Coxeter system with n generators. Construct the reflection representation of (W, S) on V. By the proof of theorem III.4 this representation is irreducible, and so by remark III.2 W acts essentially on V. ■

At the end of section III.1 we described a very vague scheme by which one could extract the picture we might expect from the Tits cone of the dual of the reflection representation. We can now look at this in detail for affine and hyperbolic Coxeter groups (as mentioned *loc. cit.*). In these cases it turns out that the space is characterised by the *signature* of B.

Definition III.9. Let $\mathscr{B}$ be a symmetric bilinear form on a vector space of dimension n. The **signature** of $\mathscr{B}$ is the triple (n_+, n_0, n_-), where n_+, n_0 and n_- are the sums of the dimensions of the positive, null, and negative eigenspaces respectively of the matrix associated to $\mathscr{B}$. Clearly each of n_+, n_0 and n_- is a non-negative integer, and they sum to n.

We have already seen that W is finite if and only if the signature of B is $(n, 0, 0)$ (i.e. B is positive-definite) if and only if W acts (i.e. acts properly) on the sphere. Now suppose that W is an irreducible infinite reflection group in the sense of the first chapter, that is to say, an affine reflection group. Geometrically then, it acts essentially on a (Euclidean) inner product space V' of dimension $n-1$, where n is the number of generators, since every chamber is a simplex with n faces, so has dimension $n-1$. The reflections in the walls of a chosen fundamental chamber C are generators of W, which we shall call S, let $\{e'_1, ..., e'_n\}$ be the canonically chosen unit normals to the walls of C, then there are positive real numbers c_i such that $\sum_i c_i e'_i = 0$, and up to scaling, this is the only non-trivial zero linear combination of the unit normals (see I.1). In particular we can choose them such that $\sum_i c_i = 1$.

Then we can choose V, an n–dimensional vector space with basis $\{e_1, ..., e_n\}$, such that V' is a co-dimension 1 subspace in which the orthogonal projection of each e_i is e'_i. Then the reflection representation of W on V has associated bilinear form B which restricts to the inner product on V', and has null space $V^\perp$ generated by $v = \sum_i c_i e_i$, hence the signature of B is $(n-1, 1, 0)$. By the way B is defined, we can conclude that this is the signature for B when we construct the reflection representation on any real n–dimensional vector space V.

Now, given a reflection representation of (W, S) on V, let v span the null space of B, then B restricts to a positive-definite form B' on $V' := V/v\mathbb{R}$. The fixed point set of the W action on V is the intersection of the hyperplanes $B(e_s, \cdot) = 0$, i.e. $V^\perp$, so W fixes v. We conclude that W leaves invariant the hyperplane $E_0 = \{f \in V^* \mid f(v) = 0\} \subseteq V^*$, which is the hyperplane in V^* orthogonal to $V^\perp$. E_0 can be identified with the dual of $V/v\mathbb{R}$, and so with $V/v\mathbb{R}$ itself since B' is positive-definite. We can call the corresponding positive-definite form on E_0, B''. Finally we define the affine hyperplane in V^* which is parallel to E_0 by $E = \{f \in V^* \mid f(v) = 1\}$, which inherits the inner product structure of E_0. By its definition, it is W–invariant. It intersects the Tits cone, and W acts on it as an affine reflection group where the walls are $wH_s \cap E$, and the chambers are $wC \cap E$ (these claims are rigorously justified in [1, section 10.2.]). We have already seen this construction in the simplest case of the infinite dihedral group in the proof of lemma III.2, where E was called the line L, see figure III.1.

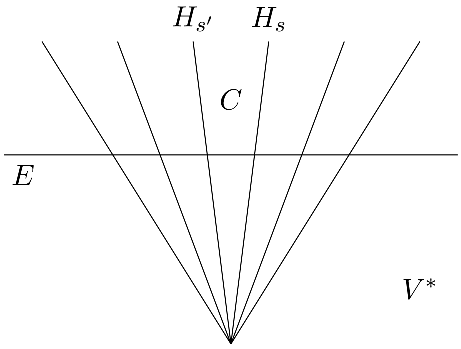


Figure III.1: The Tits cone for D_∞, by restricting the action of W to the affine line E, we recover the picture we saw in example I.5.

We shall treat hyperbolic Coxeter groups even more sketchily now that we have seen the idea in practice. Indeed, anyone who has seen the construction of $(n-1)$–dimensional hyperbolic space as one sheet of the hyperboloid in $\mathbb{R}^{n-1,1}$ will readily understand what we must do as soon as we give the definition of a hyperbolic Coxeter group.

Definition III.10. A Coxeter system (W, S) is **hyperbolic** of the associated bilinear form has signature $(n-1, 0, 1)$, and $B(x, x) < 0$ for all $x \in C$, where C is the fundamental chamber of the reflection representation on V. We then also say that W is hyperbolic.

The orthogonal group with respect to a symmetric bilinear form with this signature stabilises a hyperboloid of two sheets in V, with axis the negative eigenspace. One of these sheets, with the induced metric from B, is a model for $(n-1)$–dimensional hyperbolic space. If we intersect the Tits cone of W with the corresponding hyper-surface in V^*, we get a W–invariant $(n-1)$–dimensional geometry on which W acts essentially, and in which it is generated by reflections in hyperbolic lines. The notions of walls and chambers go through. Just as with the finite Coxeter groups, there are classifications of the irreducible affine Coxeter groups, as well as the irreducible hyperbolic reflection groups which have compact fundamental chamber (see appendix A).

III.4 Combinatorial Reflection Groups Re-imagined

In chapter II we proved the equivalence between Coxeter groups and combinatorial reflection groups. In principle this is enough on its own to justify applying the results in chapter I to Coxeter groups (at least of finite or affine type). Nevertheless we have gone to all the trouble of defining the reflection representation, and showing that it has the very nice properties we require — why do this? The reason is that the definition of combinatorial reflection groups is still combinatorial, that is the walls and chambers are abstract sets on which W must act in a way which is complicated to write down (q.v. (II.2)). To re-write the theory of chapter I in this context gets quite messy, as was seen in section 3D of chapter II. The reflection representation is a much nicer way to realise combinatorial reflection groups as geometric reflection groups.

With theorem III.2 we have a new way of thinking about Coxeter groups using the language and theory of chambers and galleries as laid out in III.2. We shall use this new interpretation to recast what we have looked at in this chapter in a new light. We began by considering combinatorial group theory in general, and in particular three decision problems. We have already seen that the Word Problem has a very easily stated geometric solution. The Conjugacy Problem is difficult for any group, combinatorial or otherwise, so one may not necessarily expect geometry to help. We know that the Conjugacy Problem for Coxeter groups *is* solvable [22]. We do not know of a solution to the Isomorphism Problem, although the reflection representation cannot help directly, because the construction is dependent on the choice of presentation.

The main bulk of the previous chapter was spent trying to prove three combinatorial results. This required 10 other lemmata, propositions, and theorems, in addition to the three main results, and a lot of heavy work with the length function. We shall now prove these three results with considerably less effort using geometric arguments from chapter I, recalling that the results cited from the finite reflection group carried over to the infinite case.

Proposition A. *Let (W, S) be a Coxeter system, and let $w \in W$. Then $l(sw) = l(w) \pm 1$ for all $s \in S$.*

Proof. This follows from the correspondence between minimal galleries and minimal expressions, as well as step 2 of the proof of theorem I.1. Alternatively it is a special case of proposition I.1(*3*). ■

Proposition B. *Let (W, S) be a Coxeter system with length function l. W is finite if and only if there is a unique longest element in W with respect to l.*

Proof. The proof of step 5 of the proof of theorem I.1 shows that the walls crossed by a minimal gallery from the fundamental chamber C are distinct. Since the length of such a gallery is equal to the number of walls it crosses, which is equal to the length of the corresponding word, longest elements of W correspond to chambers of X which are on the opposite side of every wall in $\mathcal{H}$ from C. Since $\mathcal{H}$ is finite, such chambers exist. We claim that the chamber is unique. As we have been doing, let C be the intersection of all of the "positive" half-spaces with respect to $\mathcal{H}$. Then by the description above, these chambers must lie in the intersection of all of the "negative" half-spaces with respect to $\mathcal{H}$. By definitions I.5 and I.7, this intersection consists of exactly one chamber, hence uniqueness.

The reverse direction is clear, since a longest element implies a finite number of chambers and walls, hence the group contains a finite number of reflections, and so the is itself finite. ■

Remark III.4. The diameter of X, as defined in definition I.11, is precisely the length of the unique longest word, see example I.2 and 3 of example II.4.

Proposition C. *A group W generated by a set S is a Coxeter group if and only if it satisfies the deletion condition* ***(D)****:*

> *If $w \in W$ is represented by the word $(t_1, ..., t_d)$ with $d > l(w)$, then there are indices $i < j$ such that $w = t_1 \cdots \hat{t}_i \cdots \hat{t}_j \cdots t_d$.*

where $\hat{t}_i$ indicates that that letter has been deleted from the expression.

Proof. For the forward implication, suppose $(t_1, ..., t_d)$ is a non-minimal expression for w, then the gallery

$$\Gamma : C, t_1C, t_1t_2C, ..., t_1t_2 \cdots t_dC$$

crosses a wall more than once, since if it only crossed each of the walls separating C and wC once, it must cross a wall which does not separate them, and so must cross that wall again. Then the argument which gave a contradiction in the proof of step 5 of the proof of theorem I.1, shows that two letters can be deleted from the word.

We shall not provide a geometric proof of the other implication as we feel that the point has been made, and it would require a slightly deeper analysis of X. Schematically however, it goes as follows: if we are happy to take the proof that **(D)**, **(E)**, and **(F)** are equivalent as sufficiently straightforward, we use **(F)** to characterise which posets are possible for X to correspond to a geometric reflection group, and hence a Coxeter group; for details see [8, chapter III, section 4]. ■

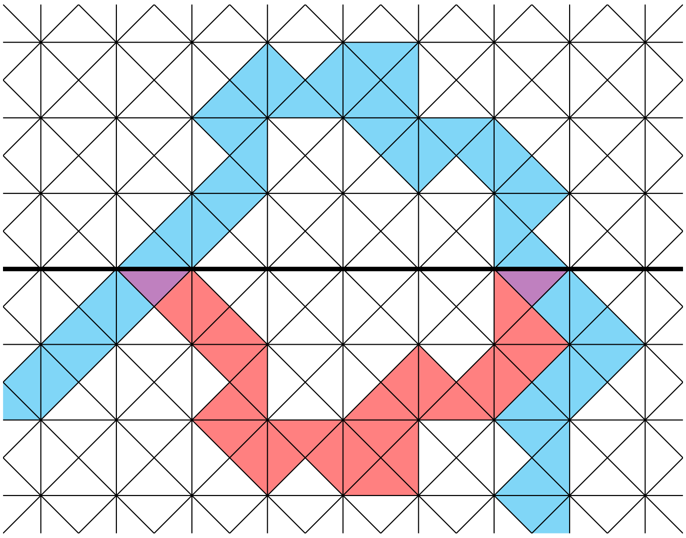


Figure III.2: Visualisation of the deletion condition using galleries. Part of a gallery (blue) is shown, which crosses a wall (thick black line) twice. The loop of the gallery cut off by this wall is reflected (red) and the new gallery stutters in two places (purple).

Remark III.5. Intuitively we can see what is going on in the proof of the deletion condition. The non-minimal gallery defines a path which meanders through V, crossing a wall H twice. So H cuts off a loop of this path. If we reflect this loop in the H, chambers are taken to chambers (see paragraph 1 of section 1D), so we get another gallery. Since chambers which were adjacent along H and distinct in the original gallery are mapped to the same chamber on just one side of H in the new gallery, the new gallery stutters in two places, so we can shorten it by removing two chambers which meet H. This is illustrated in figure III.2. We could imagine doing this repeatedly for every wall a gallery crossed multiple times to eventually end up with a minimal gallery. This is in fact the basis of the combinatorial solution to the word problem mentioned in section III.2.

Many of the other combinatorial results which we proved in section II.3 can also be proved using geometric arguments such as these. We leave it as an exercise to prove lemma II.5 in this way, since the combinatorial proof was omitted.

We shall make brief mention of a lexical oddity which may have caught your attention. We introduced two conditions on a combinatorial group, the exchange condition **(E)**, and the strong exchange condition **(SE)**. As a reminder, they were as follows:

> Let $(t_1, ..., t_d)$ be a minimal expression for $w \in W$, and let $s \in S$. If $l(sw) = l(w) - 1$, then there is some index i such that $sw = t_1 \cdots \hat{t}_i \cdots t_k$.

and

> Let $(t_1, ..., t_k)$ be an expression for w, and let $r \in R$. If $l(rw) < l(w)$, then there is some index i such that $rw = t_1 \cdots \hat{t}_i \cdots t_k$.

It is clear from the statements that they are closely related, and indeed the second seems a stronger statement than the first: the expression need not be minimal, and it works for any reflection, not just the generators[15]. However, in the proof of proposition C, we proved that they are equivalent. Why should this be the case? Again we can use geometry. The set of reflections corresponds to the set of walls $\mathcal{H}$. When we chose generators of our geometric reflection group,

[15] J. E. Humphreys meets in the middle, stating **(E)** without the minimality condition [17, section 1.7, remark].

we chose the walls of a fixed chamber. But the choice of this chamber was essentially arbitrary, and there is nothing special about one such subset of $\mathcal{H}$ over another. This means that many results about the generators S can be generalised to all reflections R, for example lemmata II.4 and II.5, see also remark II.1. The reason the expression need not be minimal is given in the preceding remark.

We shall finish this chapter by re-examining the observation made at the end of section I.1. We noted that the poset X for a finite geometric reflection group corresponded to a triangulation of the sphere in the vector space V. Can we see this in the construction of the reflection representation? Theorem III.4 characterises finite irreducible Coxeter systems as those for which the associated bilinear form B is positive-definite, in other words is an inner product. B can be thought of as a symmetric matrix, which (by elementary linear algebra) can be diagonalised by an orthogonal matrix. Moreover, since B is positive-definite, those diagonal entries will be strictly positive. Matrices in orthogonal group $O(n, B)$ therefore stabilise a family of n–dimensional ellipsoids in V [19, p. 429]. Since $\rho(W)$ is a subgroup of $O(n, B)$ (see proof of lemma III.7), it also stabilises these ellipsoids. If you take any one of them and intersect it with X, you will get a triangulation of that ellipsoid, which is itself homeomorphic to the $(n-1)$–sphere, so you get a triangulation of the sphere from X, just as we saw before.

Notes

1) When preparing this chapter, our main resources were [3], [6], [8], [9], [13], and [16]. [1] and [17] were very useful for section 3B. The exposition throughout is our own.
2) Many introductions to Coxeter groups skip over the significance and subtlety of the Tits cone in the infinite case, in particular that the action of W is not on the whole space V or V^*. Indeed this was a misconception we held when trying to build the complete bridge between the first two chapters, which we only fixed in the process of writing out the full justification. It is for this reason that we thought it vital it introduce the Tits cone formally, and discuss the action of W on it in detail.
3) Most books we have seen devote only a sentence or so to the fact that the faithfulness of ρ solves the word problem. We have not seen a solution phrased in the language of chambers in any of our reading.
4) The geometric proofs of propositions A, B, and C are our own, so too is the statement and proof of theorem III.6, in so far as we have not come across these in our reading.

Chapter IV

The Coxeter and Davis Complex

In the last chapter we unified the discussions we had in the first two chapters by establishing an equivalence between geometric and combinatorial reflection groups. In particular we saw how powerful the language of chambers could be in proving combinatorial results. This equivalence was achieved by studying the naturally arising representation of a Coxeter system on a particular real vector space V. To use the language of chambers, we then had to consider the poset X of cells in V which came from this representation. In this chapter we shall see how one can construct X, but skip out the construction of the reflection representation. We shall in fact be able to show that X is much more than a poset, it is a special kind of simplicial complex with useful properties called the Coxeter complex. Coxeter complexes are vital in the construction of what are called *buildings*, we shall introduce this application in section 1D.

After this we introduce another simplicial complex on which our Coxeter group W acts, and which is related to the Coxeter complex; it is called the Davis complex. In some sense it is an improvement on the Coxeter complex, in particular it admits a W–invariant piecewise Euclidean metric, which we introduce. We assume a reasonable knowledge of abstract simplicial complexes, flag complexes, and barycentric subdivision. We shall also need a certain degree of comfort with chamber complexes; everything which we use we have detailed in appendix B. We shall assume throughout this chapter that (W, S) is irreducible, unless stated otherwise.

IV.1 The Coxeter Complex

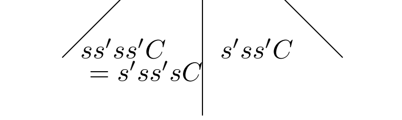
1A X as a Simplicial Complex

With the significance the poset X plays in understanding the algebraic structure of Coxeter groups, and in particular the chambers of X, it will be worthwhile exploring these more deeply. So far we have referred to X only as a *poset*; as mentioned above however, it is in fact a simplicial complex with empty simplex (see definition B.4). This should not be much of a surprise in light of our results on the structures of chambers (theorems I.2 and I.3). In the case of an infinite group acting essentially, the chambers are simplicies, and it follows immediately that X is a simplicial complex. In the case that we have a finite group acting essentially, the chambers are simplicial cones. We can get a simplicial complex from this by intersecting X with an $(n-1)$–sphere centred on the cone-point as discussed at the end of section I.1. The cells of X are in one-to-one correspondence with the simplicies of this complex of dimension one less, with the exception of the cone-point itself, however it is straightforward to recover X from this simplicial complex by adding a single point disjoint from the complex and taking the cone over the complex, so we lose no information. Moreover the notions of chambers, walls, and galleries are unchanged, just as in the infinite case. Another way to achieve the same effect is to identify the cone-point with the empty simplex, and view rays as points, sectors as edges and so on. This is in fact what we shall do. Throughout this section (until IV.2), whenever we say simplicial complex, we mean simplicial complex with empty simplex.

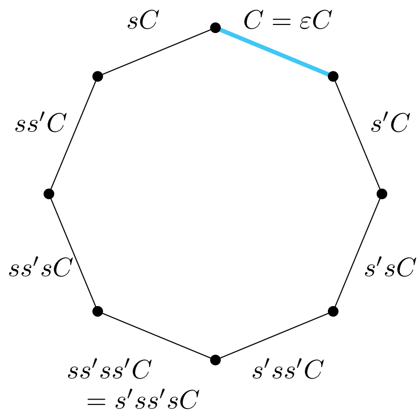
Definition IV.1. Let (W, S) be a Coxeter system acting essentially on a vector space V, and let X be the associated poset of cells. The **Coxeter complex** of (W, S), denoted $X(W, S) = X$, is the simplicial complex obtained from X as described above.

Remark IV.1. It is difficult to draw a simplicial complex and show the empty simplex. On the one hand we could draw it as a cone, where n–simplices are represented by the the positive linear span of an n–simplex (i.e as a cone), and thus as an $(n+1)$–dimensional object; or else omit to draw the empty simplex, and represent simplices as one would expect, and keep in mind that there is a hidden part of the simplicial complex in the background. We have opted for the latter. We recommend thinking of the empty simplex as a cone-point *outside* the space in which the simplicial complex exists. This coincides with our experience with the reflection representation, where the origin was the cone-point, but we could restrict the action of $\rho^*(W)$ to some co-dimension 1 hyper-surface, in which the Tits cone restricted to a simplicial complex.

Let us see this in the case of D_4, which we looked at in example I.2. The picture is as in figure IV.1. D_4 acts as the symmetry group of the square, however in this example we have constructed X as the boundary of an octagon, the connection is that the octagon is the barycentric subdivision of the square (see appendix B.4). This is typically what we get, as we shall see. In the first chapter we also saw the example of the dodecahedron (see example I.4), it is not too surprising that the Coxeter complex in this case is the barycentric subdivision of the tiling of the sphere by pentagons as shown in figure I.5b.



(a) The poset X.



(b) The simplicial complex X.

Figure IV.1: The poset X in the case of D_4 is restricted to a simplicial complex.

We shall look more closely at the chambers of X, given the significant role they played in previous chapters, in particular with respect to the definitions in appendices B.6–B.8.

Proposition IV.1. *Let (W, S) be a Coxeter system, then the associated Coxeter complex X is colourable.*

Proof. Let C be the fundamental chamber in X, and colour its vertices using a set I. Since W acts simply transitively on X (theorem I.1) every simplex A in X can be identified with a unique facet of C. Colour the vertices of A the same as its image under this identification. It is clear that this defines a valid colouring of X. Formally it follows by a simple induction on the length of the minimal gallery joining C to any other chamber. (Adapted from [8, chapter I, appendix C, proposition]) ■

All of our applications of the poset X (which we now think of as a simplicial complex), to the combinatorics of Coxeter systems used only ideas of adjacency and chambers, and by extension galleries. The vertices and other lower–dimensional cells did not really play a role. By using i-adjacency (definition B.17) arising from a colouring of X, we can show that we need only consider the chamber system associated to X. X always satisfies the conditions of lemma B.2, which guarantees this.

Our discussion in chapter I tells us that X is always thin, which is to say every chamber has exactly two adjacent chambers with respect to a given wall (including itself). The following result will also be useful to us.

Proposition IV.2. *Let X be the Coxeter complex associated to some Coxeter system, and let A be a simplex in X. Then if D and D' are two chambers which have A as a facet, every chamber in every minimal gallery connecting D and D' has A as a facet. It follows since X is a chamber complex, that lkA is also a chamber complex.*

Proof. Wlog assume D is in the positive half-space of all of the walls in X. Let $\Gamma : D = D_0, ..., D_d = D'$ be such a minimal gallery. By proposition I.1 the walls $H_1, ..., H_d$ which Γ crosses separate D and D'. For each i we have that $A \subseteq \overline{D} \subseteq H_i^+$, and $A \subseteq \overline{D'} \subseteq H_i^-$, and hence $A \subseteq H_i$. We shall prove the claim inductively: we know that $D = D_0$ has A as a facet, assume that so do $D_1, ..., D_{i-1}$. But then $A \subseteq \overline{D}_{i-1} \cap H_i = \overline{D}_{i-1} \cap \overline{D}_i$, and so A is a facet of D_i as required. [8, chapter 1, appendix D, proposition 2] ■

1B A Combinatorial Definition

As promised at the start of this chapter, we shall now give a way to construct X for a combinatorial reflection group without using the reflection representation. That is to say, given nothing more that the combinatorial group presentation (or equivalently the Coxeter matrix or diagram), we shall be able to construct an abstract simplicial complex which is isomorphic to X as a poset, and which we shall therefore also call X. We recall from chapter II section 2C the definition of a special subgroup:

Definition IV.2. Let (W, S) be a Coxeter system, and let T be a subset of S. We write W_T for the subgroup $\langle T \rangle$ of W generated by T (i.e. the group generated by T with all the relations of W which use only letters of T). Such subgroups are called **special subgroups**, and then T is called a **special subset** of S.

We further define:

Definition IV.3. If W_T is a special subgroup of a Coxeter system (W, S), for any $w \in W$, the coset wW_T of W_T is called a **special coset**[1].

The collection of special cosets of W form a poset under set inclusion. We shall however consider them with the opposite ordering, that is we shall say that

$$wW_T \leq w'W_{T'} \Longleftrightarrow wW_T \supseteq w'W_{T'}$$

for any $w, w' \in W$ and $T, T' \subseteq S$. This is certainly a poset; we shall now verify that it is isomorphic to the poset of cells of the simplicial complex X. We shall do this by constructing

[1]Not to be confused with parabolic subgroups, which are *conjugates* of special subgroups, and are often mentioned in the literature.

an isomorphism ϕ_C between the subcomplex $X_{\leq C}$ of facets of a fundamental chamber C and the special *subgroups* of W. Since W acts simple-transitively on the chambers of C, and hence simply-transitively on the W–equivalent cells of X, this isomorphism will immediately extend to a poset isomorphism ϕ from the special *cosets* to the whole of X.

We shall do this in two steps: first we shall show that the special subgroups are the poset of cells of a simplicial complex, and then we shall show that that simplicial complex is isomorphic to $X_{\leq C}$. Let S be our vertex set, then each subset T of S corresponds to a simplex of dimension 1 less than its cardinality. We shall therefore take as our simplicial complex the power set $\mathcal{P}(S)$ of S. It is clear that $W_T \subseteq W_{T'}$ if and only if $T \subseteq T'$, so taking the opposite orderings on both, $W_T \geq W_{T'}$ if and only if $T \geq T'$. Since $W_T = W_{T'}$ if and only if $T = T'$, the correspondence $T \mapsto W_T$ is a poset isomorphism between the cells of $\mathcal{P}(S)$ and the special subgroups of W.

Since C is a simplex (noting that a simplicial cone is now being thought of as a simplex by identifying the cone-point with the empty simplex), its cells are the power set of its vertices, so C and $\mathcal{P}(S)$ are isomorphic as simplicial complexes if they have the same number of vertices, but we know this to be true by the way X was constructed. We want to be slightly more clever, because we want W to act on the special subgroups in the same way as it acts on X so that we can extend this isomorphism on $X_{\leq C}$ to the whole of X. It is sufficient to define ϕ_C on the vertices of $X_{\leq C}$. The vertices are exactly the relative interiors of the subsets of $\overline{C}$ which are stabilised by all but one of the walls of C (the subset stabilised by all of the walls corresponds to the empty simplex). In particular a vertex of C can be written

$$int\left(\bigcap_{s\in S\setminus\{s_i\}} H_s\right)$$

for $i = 1, ..., n$. Hence there is a natural choice for ϕ_C on the vertices

$$\phi_C\left(int\left(\bigcap_{s\in S\setminus\{s_i\}} H_s\right)\right) = W_{\{s_i\}}$$

By characterising each facet of C by the special subgroup of W which stabilises it, this extends to $X_{\leq C}$ as

$$\phi_C\left(int\left(\bigcap_{t\in T} H_t\right)\right) = W_T.$$

From this, we can define an action of W on the special cosets of W via $w(w'W_T) = (ww')W_T$, and use this to extend ϕ_C to ϕ by demanding that ϕ commutes with the action of W, that is, define

$$\phi\left(w.int\left(\bigcap_{t\in T} H_t\right)\right) = w.\phi_C\left(int\left(\bigcap_{t\in T} H_t\right)\right) = wW_T.$$

As mentioned above, the simple-transitivity of the W action means that this is well-defined. The reason why we needed to choose the opposite ordering on the special cosets is now clear: the dimension of the faces of C vary inversely to the number of walls of C which stabilise them.

To summarise what we have done, we shall give another way of looking at this construction. Given a Coxeter system, compute its reflection representation on V (or V^*), and let X be the simplicial complex of cells in V (or V^*) obtained by identifying the cone-point at the origin with the empty simplex. We can label the facets of the fundamental chamber of X by the special subgroup of W which stabilises them, and use the W action to extend this to a labelling of the whole of X, in the same way that we initially labelled the chambers of X by the elements of W. This labelling has the same poset structure as the poset of cells of X, and by considering the special subsets corresponding to the special subgroups, we can obtain a description of this poset as a simplicial complex, where the vertex set of the fundamental chamber is exactly the set S. This means that we can forget about the reflection representation, and V, and construct X merely by calculating the special cosets of W. Moreover, the natural W action on X is preserved in this new construction. There is no harm therefore in retaining the same notation as before and defining:

Definition IV.4. The simplicial complex defined above, whose poset of cells is the set of special cosets of (W, S), is the **Coxeter complex**[2], denoted $X(W, S) = X$.

From our previous discussion, we know that X is a thin colourable chamber complex determined up to isomorphism by its chamber system. This new construction gives a naturally colouring of X by S as $\kappa(wW_{S\setminus\{s_i\}}) = s_i$, and so each simplex wW_T has its vertices coloured by the set $S\setminus T$. We noted above that all colourings were essentially the same, however what makes this choice nice is that the action of W is type-preserving (see definition B.16). Moreover, two adjacent chambers D and D' are s–adjacent in the sense of definition B.17 if and only if $D = wC$ and $D' = wsC$. We shall call this colouring the **canonical colouring** of X, and henceforth κ will refer to this colouring. This definition also coincides with our labelling of chambers, since under ϕ, the chamber we were formerly calling wC has become $wW_\emptyset = w\{\varepsilon\} = \{w\}$. We should also note that the empty simplex in X satisfies $\phi[\,] = W_S = W$, i.e. the whole group corresponds to the empty simplex, so we shall omit to draw it, in accordance with our discussion in remark IV.1.

Example IV.1. We saw at the start of this section how to obtain the Coxeter complex for D_4 from the geometric picture. We shall now do this by the combinatorial method. Recall that the Coxeter presentation for D_4 is

$$\langle s, s' \mid s^2 = s'^2 = (ss')^4 = \varepsilon \rangle$$

The special subgroups are $\{\varepsilon\}$, $\{\varepsilon, s\}$, $\{\varepsilon, s'\}$, and W. This last corresponds to the empty simplex, so for the purposes of drawing X we shall ignore it. The second and third contain the first as sets, so with the opposite ordering, they are the vertices of $\{\varepsilon\}$ which is a 1–simplex, and is the fundamental chamber, see figure IV.2.

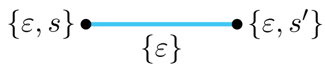


Figure IV.2: The fundamental chamber of $X(D_4, \{s, s'\})$.

The special cosets of these are

$$\{\varepsilon\}, \{s'\}, \{s's\}, \{s'ss'\}, \{ss'ss'\} = \{s'ss's\}, \{ss's\}, \{ss'\}, \{s\};$$

and

$$\{\varepsilon, s'\}, \{s', s's\}, \{s's, s'ss'\}, \{s'ss', s'ss's\} = \{s'ss', ss'ss'\},$$
$$\{ss'ss', ss's\}, \{ss's, ss'\}, \{ss', s\}, \{s, \varepsilon\}.$$

From this we get the complex X as shown in figure IV.3, which is the same as we got previously.

[2] There is no standard notation for the Coxeter complex. J. Tits [25] and K. Brown [8] use Σ, while M. Davis [13] and P. Bahls [3] use this for the Davis complex. We have opted to follow [3] where X is used for the Coxeter complex. It is as fitting a symbol as any other, as it is redolent of the conic structure of the geometric definition.

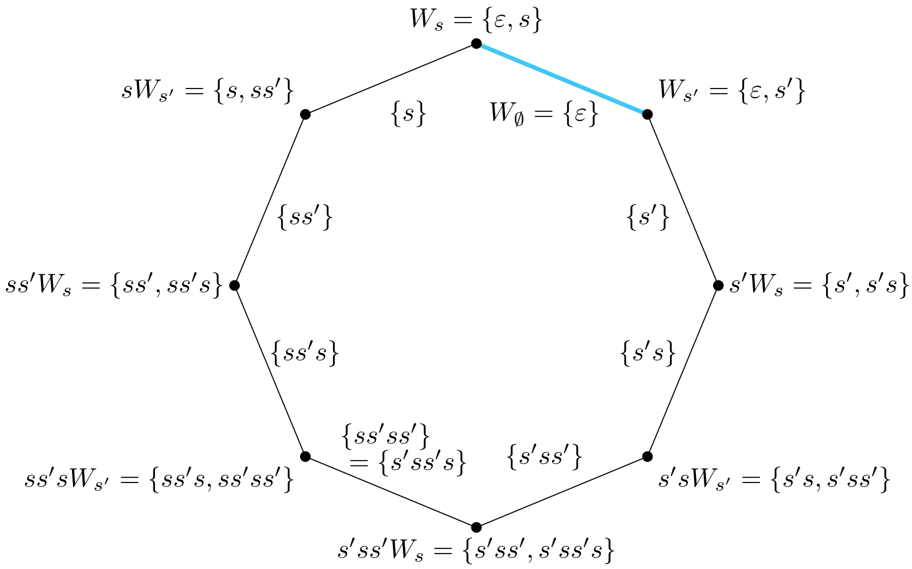


Figure IV.3: The combinatorial Coxeter complex $X(D_4, \{s, s'\})$.

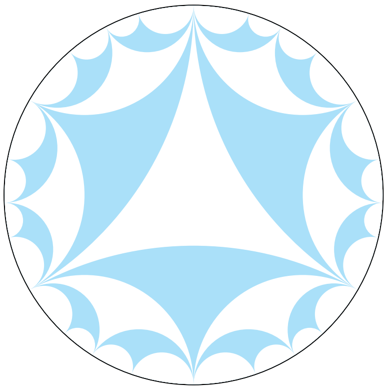
1C Properties of the Coxeter Complex

We have thus achieved our goal of constructing a simplicial complex on which W acts naturally, while circumventing the reflection representation. This construction, although it looks a lot more abstract and complicated, is far more practical to compute than via the reflection representation. Either way however, it is clear that the dimension of X is $\#S - 1$, so drawing it becomes tricky even when there are three generators (for example in the finite case, we know that X triangulates S^2, so X lives in $\mathbb{R}^3$). and so the only really practical examples to do explicitly as above are the dihedral groups, and maybe A_3, for which X is the barycentric subdivision of the boundary of a 3–simplex (a tetrahedron).

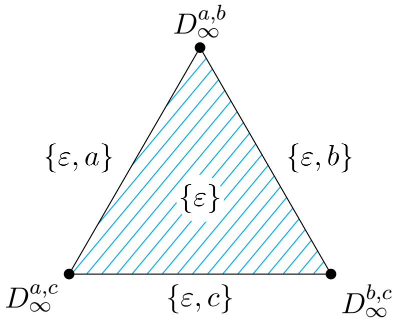
We shall not stop here however. We shall prove some properties of Coxeter complexes, and then go on to discuss their main application, which is to so-called buildings. We have noted that the few finite examples which we have considered are barycentric subdivisions. In particular we have seen that the Coxeter complexes of a couple of the symmetry groups of regular polytopes are the barycentric subdivisions of the boundaries of those polytopes. The reason why is quite simple: since they are regular, every vertex, edge, and face *et cetera* has a symmetry hyperplane passing through the middle of it, so the symmetries halve up the polytope in every possible way, which corresponds to taking the barycentric subdivision. Not every finite Coxeter system is the symmetry group of a regular polytope (for example E_6, E_7, and E_8), however in general we can prove this slightly weaker result (weaker because every barycentric subdivision is a flag complex).

Proposition IV.3. *The Coxeter complex X associated to a finite Coxeter system is a flag complex (see definition B.11).*

Proof. We shall show that X satisfies statement (2) of proposition B.1, that every finite family of pairwise joinable simplicies is joinable, which is equivalent to X being a flag complex. Since X is finite, the set of walls $\mathcal{H}$ is also finite. Since each $H \in \mathcal{H}$ is of co-dimension 1, H cannot strictly separate two joinable simplicies in X, so a finite family of pairwise joinable simplices must lie in a closed half-space $\overline{H^*}$ for each H. Then the intersection $\bigcap_{H \in \mathcal{H}} \overline{H^*}$ is a closed cell in X which contains the whole family. In particular the interior of this intersection is an open cell which is an upper bound for the whole family, and hence they are joinable. [8, chapter I, appendix B, proposition 2] ■



(a) tiling of the Poincaré disc by ideal triangles.



(b) The fundamental chamber of X corresponding to the W.

Figure IV.4

One question we might ask is when X is a manifold. In the finite case, we know that X triangulates the sphere, so it is always a manifold. However not every Coxeter complex is a manifold, as seen in the following example.

Example IV.2. Consider the tiling of the hyperbolic plane by ideal triangles as shown in figure IV.4a. The symmetry group is generated by reflections in the sides of the "central" triangle, which are mutually parallel, hence

$$W = \langle a, b, c \mid a^2 = b^2 = c^2 \rangle = \triangle(\infty, \infty, \infty).$$

The special subsets are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, (and $\{a, b, c\}$ which corresponds to the empty simplex). The corresponding special subgroups are shown in figure IV.4b, which shows the fundamental chamber. $D_\infty^{x,y}$ is the infinite dihedral group generated by x and y.

The full complex X has a copy of this fundamental chamber for every ideal triangle in the tiling, and moreover, they mirror the adjacency relations between these ideal triangles (note that in the diagram, we have not drawn all of the infinite tiling up to the boundary, only a finite number of iterations). It is clear that X in this case is not a manifold, and the problem occurs at the "cusps", in particular the vertex $D_\infty^{a,b}$ is a vertex of infinitely many chambers, so X is not compact near that point.

There is however a very simple characterisation of when X *is* a manifold[3].

Theorem IV.1. *Let X be a Coxeter complex, then the following are equivalent:*

1) *X is a manifold,*
2) *X is locally finite, and*
3) *every proper special subgroup of W is finite.*

Lemma IV.1. *Let A be a simplex in X, let $T = S \backslash \kappa(A)$ be the subset of S such that A is stabilised by H_s for all $s \in T$. Then lkA is isomorphic to $X(W_T, T)$. In particular lkA is a chamber complex.*

[3]There is extremely rich theory of structures which are not quite manifolds, called orbifolds. They arise naturally as the quotient of smooth manifolds by the action of hyperbolic Coxeter groups. These spaces have corners or cusps analogous to the example above, and are vital to the modern theory of 3–manifolds. In particular they are instrumental in the proof of Thurston's Hyperbolisation Theorem, see [20] for details.

Proof. Wlog assume that A is a facet of the fundamental chamber; then A *is* the special subgroup W_T as defined. lkA is isomorphic to the poset $X_{\geq A}$ via $B \mapsto B \wedge A$ for $B \in lkA$, hence lkA is isomorphic to the poset of special cosets of W which are contained in W_T by set inclusion (the opposite of the poset ordering). These are precisely the special cosets of the Coxeter system (W_T, T), and hence lkA is isomorphic to $X(W_T, T)$ as claimed. [8, chapter 3, section 2, proposition] ■

Proof of theorem IV.1. We shall prove the contra-positive of (1) *implies* (2). Assume X is not locally finite, so there is some vertex v which is a facet of infinitely many chambers. Take a bounded closed set Y in X which contains an open neighbourhood around v and intersects only with the chambers which have v as a facet. For each of these chambers D, take an open set containing $\overline{D}$ so that there is at least one point in Y for each chamber which is contained in only one of these open sets. These open sets form an open cover of the closed set Y, but there is clearly no finite subcover, contradicting Heine-Borel, so Y is not compact, hence X is not locally compact around v, and so not a manifold.

We shall take the same approach to prove (2) *implies* (3). Suppose there is a proper special subgroup of W which is infinite, then it is the face of infinitely many chambers $\{w\}$ in X, so X is not locally finite.

Finally we prove that (3) *implies* (1) directly. Assume that all proper special subgroups of W are finite. Take a point $x \in X$, and let A be the open simplex which contains it. If $[v]$ is a vertex of A, $lk[v]$ contains neither $[v]$ nor A. By lemma IV.1, $lk[v]$ is isomorphic to the Coxeter complex of a finite Coxeter system generated by $n-1$ generators, where $\dim X = n$, so $lk[v] \simeq S^{n-1}$. The cone of $lk[v]$ over $[v]$ is a closed ball, with boundary $lk[v]$. Since $x \notin lk[v]$, x is in the interior of this ball, which is an open neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$, and hence X is a manifold. (Left as an exercise in [8, chapter III, section 2, corollary 3]) ■

1D Buildings

We motivated the definition of the Coxeter complex as a way of constructing a geometric object (a simplicial complex) on which a combinatorially defined Coxeter system acted as a group of symmetries, and in particular as an easy way of obtaining such an object without the rigmarole of the reflection representation, or worse, its dual. By far and away however, the main applications of Coxeter complexes is in the theory of buildings. Roughly put, these are highly symmetrical simplicial complexes on which certain groups act as automorphism groups. In particular they grew out of an attempt to systematically define incidence geometries relating to semi-simple groups of Lie type. They were axiomatised by J. Tits, but the history of their development is so long and convoluted, that it is not really possible to motivate the definition. More detailed historical accounts may be found in [25, introduction] and [8, chapter 5, section 4].

Definition IV.5. Let Δ be a simplicial complex, and $\mathfrak{a}$ a set of subcomplexes of Δ. The pair $(\Delta, \mathfrak{a})$ is a **(thick) building**, of which the elements of $\mathfrak{a}$ are **apartments**, if the following hold

(B1) Δ is thick (see definition B.14),

(B2) the elements of $\mathfrak{a}$ are thin chamber complexes,

(B3) any two simplices of Δ are contained in a common apartment, and

(B4) if A and A' are two simplices of Δ, both contained in each of the apartments χ and χ' of Δ, then there is an isomorphism of χ onto χ' fixing A and A'.

If the pair $(\Delta, \mathfrak{a})$ satisfies (B2)–(B4), it is called a **weak building**.

Taking A and A' to be the empty simplex in (B4) we can immediately conclude that every apartment is isomorphic. By (B3), the apartments cover Δ, so all maximal simplicies have the same dimension. Since any two chambers are contained in a common apartment, which is a chamber complex by (B2), they can be connected by a gallery, and so Δ is itself a chamber complex, and it is necessarily connected. The nomenclature surrounding buildings is attributed to N. Bourbaki by J. Tits. Think of chambers as rooms, apartments are collections of rooms connected by corridors (galleries), and buildings are blocks of apartments. It is clear that if X is a Coxeter complex, then the pair $(X, \{X\})$ is a weak building.

Example IV.3. Let X be the Coxeter complex of D_∞. X "triangulates" $\mathbb{R}$, and can be thought of as consisting of the vertices $\mathbb{Z}$, with 1–simplicies the intervals $(n, n+1)$ (see figure III.1, X is the intersection of E with the Tits cone). Let Δ be a connected infinite tree in which every vertex has valence ≥ 3, and let $\mathfrak{a}$ consist of all embeddings of X into Δ. We claim that $(\Delta, \mathfrak{a})$ is a building.

That Δ is thick is immediate from the condition on the valencies. Since each apartment is isomorphic to X, a Coxeter complex, they are thin chamber complexes. These prove (B1) and (B2). The apartments can be thought of as all infinite paths through Δ. Since Δ is connected, any two simplices are joined by a path, and hence are contained in the same apartment (this establishes (B3)). Moreover, the subset of the path which stretches between them is unique, because if not we could use two different paths to create a cycle, contradicting the assumption that Δ is a tree. This gives a sense of distance between two simplices in terms of the number of edges (chambers) which lie between them.

We can now show that (B4) is satisfied. Take two simplices A and A', and let χ and χ' be two apartments which contain both. By the observations above, they must agree between A and A', so the isomorphism between them restricted to this section is the identity. We extend this to an isomorphism to the rest of χ and χ' in analogous ways. Take the part of χ "beyond" A, then map the unique vertex a distance n from A in χ to the unique vertex a distance n beyond A in χ'. This defines a chamber map $\phi : \chi \mapsto \chi'$ defined on the whole of χ, which is the identity on A and A', and which is an isomorphism of simplicial complexes, as required. This is illustrated in figure IV.5.

If we weaken the condition on the valencies to being ≥ 2, then we have a weak building.

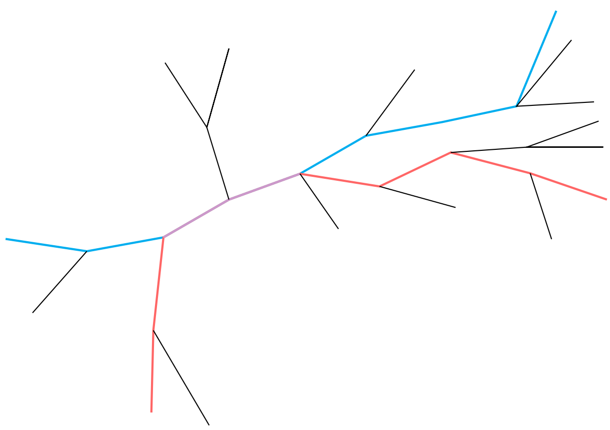


Figure IV.5: Part of an infinite tree satisfying the thick building axioms. Two apartments are drawn in red and blue, which coincide along a short branch (purple). It is clear how the isomorphism between them spreads out from this shared section.

Heuristically (B4) says that two "nearby" apartments can be identified with each other without moving then too much; one could imagine collapsing them into one apartment by pressing them together: taking two branches in the previous example and sticking them together. This might in turn bring other apartments nearby, and one could continue the process until you were left with just a single apartment. That is exactly what this next result says.

Proposition IV.4. *Let $(\Delta, \mathfrak{a})$ be a weak building, then every apartment χ is a retract of Δ.*

Proof. We shall construct a retraction of Δ onto χ analogous to the way described above. Let C be a chamber of χ. Assuming $\Delta \neq \chi$ (otherwise we are done), (B2) guarantees the existence of another apartment containing C, call it χ'. Then there is an isomorphism $\phi_{\chi',\chi} : \chi' \mapsto \chi$ which fixes C by (B4) (using the empty simplex as the other common simplex), and moreover it is unique. Indeed, since it fixes C, it fixes all of the vertices of C, and so the faces of C. Let C' be a chamber adjacent to C in χ', then the common face is fixed by $\phi_{\chi',\chi}$, and so all but one of the vertices of C' are fixed. The image the remaining vertex is now uniquely determined by the fact that $\phi_{\chi',\chi}$ is a bijection between the chambers of χ and χ'. Continuing in this way we see that $\phi_{\chi',\chi}$ is uniquely determined along all non-stuttering galleries stretching from C in χ', and since every chamber is connected to every other chamber by a gallery in a chamber complex, $\phi_{\chi',\chi}$ is uniquely determined on the whole of χ', as required.

Suppose χ'' is another apartment which contains C, then by uniqueness, $\phi_{\chi'\chi}$ and $\phi_{\chi'',\chi}$ agree on $\chi' \cap \chi''$ (we could write $\phi_{\chi''\chi} = \phi_{\chi',\chi} \circ \psi$, where $\psi : \chi'' \mapsto \chi'$ is an isomorphism which fixes $\chi' \cap \chi''$; the existence of which follows easily from (B4)).

Finally we claim that we can string together a series of these isomorphisms to get a retraction $\rho : \Delta \mapsto \chi$. Suppose χ' and χ have a chamber in common, then we can use $\phi_{\chi',\chi}$ as constructed above. If not, then let C' be a chamber in χ', and by (B3) there is an apartment χ'' containing both C and C'. As above construct $\phi_{\chi',\chi''}$ and $\phi_{\chi'',\chi}$ which fix C' and C respectively and are unique, and define $\phi_{\chi',\chi} = \phi_{\chi'',\chi} \circ \phi_{\chi',\chi''}$. Then explicitly we have that if $A \in \chi'$

$$\rho(A) = \phi_{\chi',\chi}(A)$$

which is well-defined by uniqueness. [8, chapter IV, section 3, proposition 1] ∎

Corollary IV.1. *The combinatorial distance function on a building $(\Delta, \mathfrak{a})$ coincides with the combinatorial distance function on each of its apartments. In particular the diameter of Δ is equal to the diameter of each of its apartments (see definitions I.10 and I.11).*

Proof. Let D and D' be two chambers in Δ, and let χ be an apartment which contains both (q.v. (B3)). Clearly $d_\chi(D, D') \leq d_\Delta(D, D')$. Suppose the inequality were strict, then since ρ as in the proposition is a chamber map, it takes galleries to galleries, and the image of a minimal gallery in Δ which realises $d_\Delta(D, D')$, would realise this same distance in χ, contradicting the definition of $d_\chi(D, D')$.

For the second claim, every apartment is isomorphic to every other, so the statement makes sense. We know that $diam(\chi) \leq diam(\Delta)$ for all $\chi \in \mathfrak{a}$. If it were strict, then there would be chambers D and D' both contained in an apartment χ', but with $d_\Delta(D, D') = d_{\chi'}(D, D') > diam(\chi')$ a contradiction. [8, chapter IV, section 3, corollary 1] ∎

The application of Coxeter complexes to buildings goes far beyond that fact that they are (very basic) examples of them; they are inextricably linked, as shown by the following theorem due to Tits, the proof of which relies on the above proposition and corollary.

Theorem IV.2. *Let $(\Delta, \mathfrak{a})$ be a thick building, then its apartments are Coxeter complexes. [25, theorem 3.7]*

Sketch of proof. The proof is very long and technical, but it may broadly be broken down into two steps.

Step 1: Characterise Coxeter complexes by foldings.

Formally a folding of a chamber complex Δ is a chamber map $\phi : \Delta \mapsto \Delta$ which is a retraction onto its image $\Delta' \subseteq \Delta$, such that the pre-image of every chamber C' in Δ' is $\{C, C'\}$ where $C \in \Delta \backslash \Delta'$. Heuristically a folding is exactly what it sounds like: one takes a wall, and maps the half of Δ on one side of the wall on to the half of Δ, Δ', on the other. We only had a well-defined sense of a "wall" in the case of a Coxeter complex, and not for a general chamber complex (for example, the subdivision of a pentagon into five 2–simplices is a chamber complex, but it has no walls which divide it in two). We claim that the existence of foldings characterises Coxeter complexes in the following way:

> A thin chamber complex is a Coxeter complex if and only if for every pair of adjacent chambers, there is a folding which maps one onto the other, which is left fixed.

If we have a Coxeter complex associated to (W, S), choose one of the chambers to be the fundamental chamber C, and let H_s be the wall separating it from the other chamber C', then we define ϕ on the chambers (since every Coxeter complex is determined by its chamber system, we can do this) by

$$\phi(D) = \begin{cases} D & \text{if } D \in H_s^+; \\ sD & \text{if } D \in H_s^-. \end{cases}$$

which maps C to C, and C' to $sC' = C$ as required. This is made rigorous using the folding condition **(F)**, which is why it has that name [8, chapter II, section 3B, proposition].

The other direction is more tricky. We suppose such foldings exist, and construct a set $\mathcal{H} \times \{\pm 1\}$ and a group W which acts on this set based on the foldings. Thus we show that W satisfies condition **(A)**, and is a Coxeter group, and moreover the Coxeter complex of this group is exactly the thin chamber complex on which W was made to act, which is shown by a careful analysis of the stabilisers of each of the facets. For details see [8, chapter II, section 3B, and chapter IV section 4].

Step 2: Show that the apartments of a building have the sufficient number of foldings to satisfy the criterion above.

In the proof of proposition IV.4, given a chamber C in an apartment χ we constructed what might be called the canonical retraction of Δ to χ: $\rho_{\chi,C}$. We can use this to construct a folding. Let C and C' be two distinct adjacent chambers in Δ, with common apartment χ. Since Δ is thick, the common face of C and C' is also a face of a third distinct chamber C''. Let χ' be an apartment containing C and C''. We then define $\phi : \chi \mapsto \chi$ as the restriction to χ of the retraction $\rho_{\chi,C'} \circ \rho_{\chi',C} : \Delta \mapsto \chi$, which is illustrated in figure IV.6. C is fixed by both of these retractions, so $\phi(C) = C$, what about C'? Since χ' is thin (by (B2)), $C' \notin \chi'$; $\rho_{\chi',C}$ must map it to C or C'', and since on χ it behaves as $\phi_{\chi,\chi'}$ which fixes C, and since it is an isomorphism, $\rho_{\chi',C}(C') = C''$. By the same argument, $\rho_{\chi,C'}(C'') = C$ or C', but as it fixes C', $\rho_{\chi,C'}(C'')$ must be C, hence $\phi(C') = C$ as required.

Now one uses corollary IV.1 which says that ϕ preserves distances to extrapolate its behaviour on the rest of χ, and in particular, that it satisfies the definition of a folding. In this way we have constructed a folding for every pair of adjacent chambers in χ, and so by step 1, it is a Coxeter complex. Since all apartments are isomorphic, they are all Coxeter complexes. For details, see [8, chapter IV, section 7]. ■

This remarkable result means that with our work on Coxeter complexes, we already know a lot about the structure of buildings. Since every apartment is a Coxeter complex corresponding to the same Coxeter system (W, S), we can say that Δ is of type (W, S), and associate to it a Coxeter matrix and diagram. We noted in the last section that Coxeter complexes are colourable; this is a property inherited by buildings.

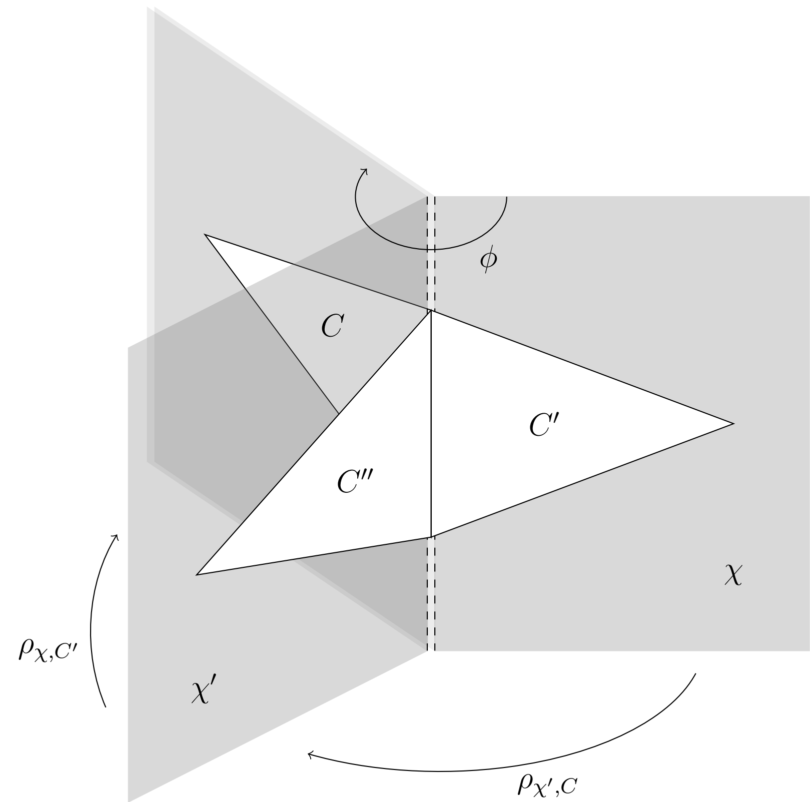


Figure IV.6: The three chambers C, C', and C'' in the apartments χ and χ'. The action of $\rho_{\chi,C'}$ and $\rho_{\chi',C}$ are indicated by arrows.

Proposition IV.5. *Any building $(\Delta, \mathfrak{a})$ is colourable, and the isomorphisms in (B4) can be chosen to be type-preserving.*

Proof. Fix a chamber C and colour its vertices. By (B3) C is in an apartment χ, which is a Coxeter complex, and hence we can extend this to a colouring, κ, of the whole of χ by proposition IV.1. Now if χ' is another apartment containing C, the colouring κ' of χ' obtained in the same way agrees on $\chi \cap \chi'$, since it is uniquely determined by the colouring of C, and they are isomorphic. Therefore the colourings of all the apartments containing C match up to give a colouring of the union of these apartments, which, by (B3), is the whole of Δ.

Now suppose χ and χ' share a chamber and a simplex, then the isomorphism ϕ in (B4) fixes this chamber so is automatically type-preserving. If we have two apartments χ and χ' which share only two simplicies A and B, we can choose chambers $C \in \chi$ and $D \in \chi'$ which have A and B as faces respectively. Let χ'' be an apartment containing C and D. The isomorphisms $\chi \mapsto \chi''$ and $\chi'' \mapsto \chi'$ are type-preserving by the first case, and their composition is a type-preserving isomorphism $\chi \mapsto \chi'$, as required. ∎

The theory of buildings is extremely rich and deep, and what we have mentioned here does not scratch the surface. They are closely related to classical incidence geometries, indeed in [8, chapter IV, section 2], K. Brown considers low rank cases of buildings of dihedral type (their apartments are the Coxeter complexes of dihedral groups), and the corresponding buildings are the flag complexes of a 2–dimensional incidence geometry, the projective plane, polar geometry *et cetera.* They can also be related to abstract algebra via so-called Tits systems, or BN-pairs, which are systems of subgroups of a given group G satisfying a set of axioms. These readily give examples of buildings, and again are closely related to Coxeter systems. They are discussed in [1], [6], and [8].

IV.2 The Davis Complex

The aim of this section is to construct the Davis complex, and look at some of its properties. The purpose of the Davis complex is to associate to an combinatorial Coxeter system, a geometric object on which the group acts naturally as a group of symmetries. In particular, the Davis complex can be constructed just from the group presentation. This may sound very familiar from the opening of the previous section, where we justified the study of Coxeter complexes. The Davis complex is a different, but related construction to the Coxeter complex, due to M. Davis, but studied extensively by G. Moussong (and others), so some sources refer to it as the Moussong complex. In the introduction to his book [13], M. Davis notes two drawbacks to the otherwise extremely useful Coxeter complex: (a) that the fundamental domain for the action of W on X, the fundamental chamber C, is not compact, and (b) there is no natural W–invariant metric on X.

The Davis complex is a simplicial complex (without empty simplex) on which W acts "properly" as a group of poset automorphisms generated by reflections; moreover, the fundamental domain for the group action is compact, and the natural piecewise Euclidean metric is W–invariant (W acts by isometries) and is so-called CAT(0). The Davis complex also embeds in a natural way into $\mathcal{I}$, the interior of the Tits cone of (W, S) (see (III.1) on page 43). There are also relations to buildings. Throughout the rest of this chapter, when we say simplicial complex, we mean simplicial complex *without* empty simplex.

2A The Nerve of a Coxeter System

Recall that for the combinatorial definition of the Coxeter complex, we considered special subgroups and special cosets. For the Davis complex we are slightly more discriminating.

Definition IV.6. Given a Coxeter system (W, S), we say $T \subseteq S$ is **spherical**[4] if the subgroup generated by T, W_T, is finite; if this is the case W_T is called a **spherical subgroup**. We know then that (W_T, T) is also a Coxeter system. Denote by $\mathcal{S} = \mathcal{S}(W, S)$ the set of a **spherical subsets** of S, this is clearly a poset (see appendix B.1) ordered by inclusion. Write $\mathcal{S}^{(k)}$ for the set of of spherical subsets of cardinality k.

We have already considered these subgroups when we defined the interior of the Tits' cone (see (III.1)). Since each generator in a Coxeter system is an involution, W_s for any $s \in S$ is the finite group of order 2, so $S \subseteq \mathcal{S}$. Moreover, if T generates a finite Coxeter group, so too does any subset of T, the two conditions for being an abstract simplicial complex are satisfied so long as we exclude the empty set, hence $\mathcal{S}_{>\emptyset}$ is an abstract simplicial complex.

Definition IV.7. The **nerve** of a Coxeter system (W, S) is the abstract simplicial complex $\mathcal{S}_{>\emptyset}$ which we denote by $L = L(W, S)$.

It follows that $\mathcal{S}^{(k)}$ is the set of $(k-1)$–simplicies in L, so $\bigcup_{i=1}^{k} \mathcal{S}^{(i)}$ is the $(k-1)$-skeleton of L. Some simple examples illustrate this definition.

Example IV.4.

1) If (W, S) is finite to begin with, we have $\mathcal{S} = \mathcal{P}(S)$, the power set of S, and so L is the full simplex on S.
2) Take $W = D_\infty = \langle a, b | a^2 = b^2 = \varepsilon \rangle$, then $S = \{a, b\}$, and $\mathcal{S} = \{\emptyset, \{a\}, \{b\}\}$, so L consists of just two points, i.e. the 0-sphere S^0.

[4]This name comes from the fact that finite Coxeter systems are characterised by the fact that they "act on the sphere", see last paragraph of chapter III.

3) If (W,S) is reducible, such that $(W,S) = (W_1 \times W_2, S_1 \sqcup S_2)$. It is easy to check that $T = T_1 \sqcup T_2$ is spherical in (W,S) if and only if T_1 and T_2 are spherical in (W_1,S_1) and (W_2,S_2) respectively, hence

$$\mathcal{S}(W,S) = \mathcal{S}(W_1,S_1) \times \mathcal{S}(W_2,S_2)$$

and

$$L(W,S) = L(W_1,S_1) \vee L(W_2,S_2)$$

where $\vee$ is the join, see definition B.7.

Taken from [13, examples 7.1.2, 3, and 5].

If we have a Coxeter system (W,S), we have seen that we can express it using a Coxeter diagram $\mathcal{V}$. By convention we omitted edges labelled 2 and included edges labelled ∞ because this gave us the result that a Coxeter system was irreducible if and only if its Coxeter diagram was connected (see lemma II.1). If instead we define the graph $\tilde{\mathcal{V}}$ to be as $\mathcal{V}$, but where the edges with label 2 were included, and the edges labelled ∞ were excluded, then it is clear that the 1–skeleton of $L(W,S)$ is precisely $\tilde{\mathcal{V}}$ [13, example 7.1.6].

2B The Davis Complex and the Fundamental Chamber

Definition IV.8. A **spherical coset** is a coset of a spherical subgroup in W, i.e. wW_T for some $w \in W$, $T \subseteq S$, such that $\#W_T < \infty$. The set of all spherical cosets in W is written

$$W\mathcal{S} = \bigcup_{T \in \mathcal{S}} W/W_T$$

which is a poset under set inclusion.

There is a well-defined projection $W\mathcal{S} \twoheadrightarrow \mathcal{S} : wW_T \mapsto T$ and a natural inclusion map $\mathcal{S} \hookrightarrow W\mathcal{S} : T \mapsto W_T$. W acts naturally on $W\mathcal{S}$ via $w.(w'W_T) = (ww')W_T$. $\mathcal{S}$ and $W\mathcal{S}$ are both posets ordered by inclusion. Every abstract simplicial complex is a poset ordered by inclusion, but the reverse does not hold. There is a very simple way of obtaining an abstract simplicial complex from an arbitrary poset, using what are called flags, see appendix B.5.

Now, going back to $\mathcal{S}$ and $W\mathcal{S}$, we denote the geometric realisation[5] of the poset $\mathcal{S}$ to be $|\mathcal{S}| = K = K(W,S)$ (note the differences between this and the nerve L which is $\mathcal{S}_{>\emptyset}$), and the geometric realisation of the poset $W\mathcal{S}$ to be $|W\mathcal{S}| = \Sigma = \Sigma(W,S)$.

Definition IV.9. $\Sigma(W,S)$ is the **Davis complex** of (W,S), and $K(W,S)$ is its **fundamental chamber**[6].

The projection $W\mathcal{S} \twoheadrightarrow \mathcal{S}$ induces a simplicial projection $\pi : \Sigma \twoheadrightarrow K$, and the inclusion $\mathcal{S} \hookrightarrow W\mathcal{S}$ induces an inclusion $i : K \hookrightarrow \Sigma$. The next result says that if we identify K with its image in Σ, every simplex in Σ is a translate of a simplex in K, under the action of W on Σ, which justifies calling K the fundamental chamber.

[5] There is a easy way to go between an abstract simplicial complex (in this case the flag complex), and its geometric realisation, and so one does not need to be too worried about which one we are dealing with in general. Indeed, in the last section our definition was of an abstract simplicial complex, but we happily talked about its topological closure, meaning the closure of the geometric realisation, without comment. M. Davis gives these definitions as geometric realisations specifically, and the reason is that he will introduce a Euclidean metric on Σ, which only makes sense for the geometric realisation.

[6] M. Davis [13, p. 126] seems content to merely refer to this complex as K, although earlier (p. 64) he defines the fundamental chamber in the much more general context of a space, called $\mathcal{U}$. P. Bahls [3, p. 25] makes the direct link and refers to K as the fundamental chamber explicitly.

Lemma IV.2. *Every facet of Σ is a translate of a facet of K under the action of W.*

Proof. We can reinterpret proposition I.1(2) as saying that for every element $w \in W$, there is a well-defined subset $S(w) \subseteq S$ such that every minimal expression $(t_1, ..., t_d)$ contains only the letters of $S(w)$; it now easily follows that if $T \subseteq S$, then W_T is the subgroup of W containing all elements such that $S(w) \subseteq T$ [13, corollary 4.1.2]. Let T and T' be subsets of S, and $w, w' \in W$ then from the above observation, $wW_T \subset w'W_{T'}$ if and only if $T \subset T'$ and $wW_{T'} = w'W_{T'}$.

If $w_0W_{T_0} \subset \cdots \subset w_kW_{T_k}$, then $T_0 \subset \cdots \subset T_k$ and $w_iW_{T_i} = w_0W_{T_i}$ for all i. That is, the facet which is the flag $[w_0W_{T_0}, ..., w_kW_{T_k}]$ is the w_0 translate of the facet $[W_{T_0}, ..., W_{T_k}]$ in K. [13, lemma 7.2.3] ■

We shall now give an explicit example in detail.

Example IV.5. We have seen a lot of the example of D_4, so for some variety, consider $(W, S) = (D_3, \{s_1, s_2\})$, the Coxeter system for the dihedral group of order 6, which we know acts as the symmetry group of the regular triangle in the plane. It has presentation of Coxeter type A_2: $\langle s_1, s_2 | s_1^2 = s_2^2 = (s_1s_2)^3 = \varepsilon\rangle$. Since this is a finite Coxeter system

$$\mathcal{S} = \mathcal{P}(S) = \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}\}$$

so the nerve of (W, S) is $L = \{[s_1], [s_2], [s_1, s_2]\}$, which can be drawn as in figure IV.7.

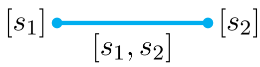


Figure IV.7: The geometric realisation of the nerve as a simplicial complex $Geom(\mathcal{S}_{>\emptyset})$.

We can consider the geometric realisation of $\mathcal{S}_{>\emptyset}$ as a *poset* by calculating its flag complex, which has n–simplicies:

0 : $[\{s_1\}], [\{s_2\}], [\{s_1, s_2\}]$

1 : $[\{s_1\}, \{s_1, s_2\}], [\{s_2\}, \{s_1, s_2\}]$

where an n–simplex is a totally ordered subset of L by inclusion, written as an $(n+1)$–tuple in square brackets starting with the minimal element. The geometric realisation of this is shown in figure IV.8.

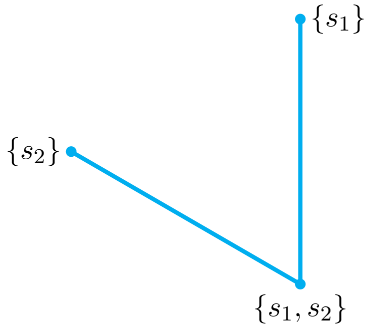


Figure IV.8: The geometric realisation of the nerve of D_3 as a poset $|\mathcal{S}_{>\emptyset}|$.

Next $Flag(\mathcal{S})$ consists of simplicies:

0 : $[\emptyset], [\{s_1\}], [\{s_2\}], [\{s_1, s_2\}]$

1 : $[\emptyset, \{s_1\}], [\emptyset, \{s_2\}], [\emptyset, \{s_1, s_2\}], [\{s_1\}, \{s_1, s_2\}], [\{s_2\}, \{s_1, s_2\}]$

2 : $[\emptyset, \{s_1\}, \{s_1, s_2\}], [\emptyset, \{s_2\}, \{s_1, s_2\}]$

the geometric realisation of which is shown in figure IV.9.

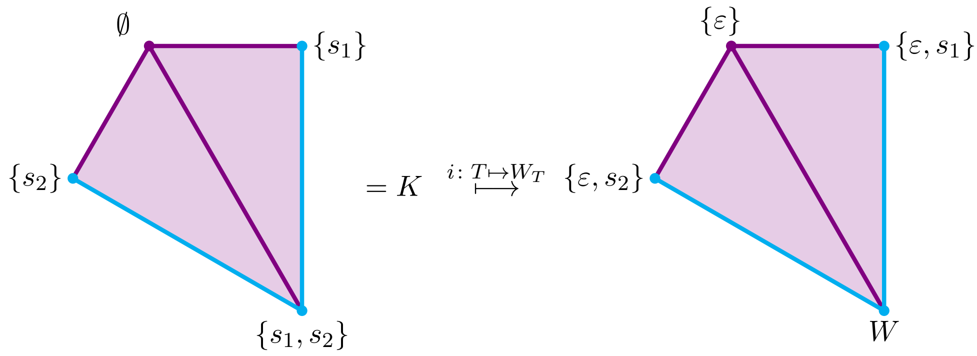


Figure IV.9: The complex K of D_3, showing the geometric realisation of the nerve as a poset. The image of K under the inclusion map into Σ is also shown.

Notice that the geometric realisation of the nerve is a subcomplex of K, and in fact K is the cone of $|L|$. Finally, we have that

$$\begin{aligned}
W\mathcal{S} &= \bigcup_{T \subseteq \mathcal{S}} W/W_T \\
&= W/W_{\emptyset} \cup W/W_{\{s_1\}} \cup W/W_{\{s_2\}} \cup W/W_{\{s_1,s_2\}} \\
&= \{\{\varepsilon\}, \{s_1\}, \{s_2\}, \{s_1s_2\}, \{s_2s_1\}, \{s_1s_2s_1\}\} \cup \\
&\qquad \{\{\varepsilon, s_1\}, \{s_2, s_2s_1\}, \{s_1s_2, s_1s_2s_1\}\} \cup \\
&\qquad \{\{\varepsilon, s_2\}, \{s_1, s_1s_2\}, \{s_2s_1, s_1s_2s_1\} \cup \\
&\qquad \{\{\varepsilon, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}\}
\end{aligned}$$

To write out every n–simplex in $Flag(W\mathcal{S})$ would be the act of a very patient person, we instead summarise that information simply by giving the geometric realisation in figure IV.10.

This has been drawn suggestively as a hexagon, because Σ is in fact the barycentric subdivision of the hexagon; however W acts on this by the symmetries of the triangle, so the W action is better shown by drawing Σ as in figure IV.11. Note how K, the fundamental chamber, sits inside Σ.

How does this relate to $X(W, S)$? Extrapolating from our example of D_4, the Coxeter complex of D_3 is the barycentric subdivision of the boundary of a triangle. Now we have constructed Σ which is the cone over the barycentric subdivision of the barycentric subdivision of the boundary of a triangle; we might write $\Sigma = \mathbf{Cone}(\mathbf{Bs}X) = \mathbf{Cone}(\mathbf{Bs}^2\Delta_{triangle})$, see definitions B.8 and B.9.

2C Properties of the Davis Complex

How does Σ relate to X? If we assume that W is finite, then the spherical subsets $\mathcal{S}$ are all the special subsets of S, and the spherical cosets $W\mathcal{S}$ coincide exactly with the special cosets. In the construction of X, we first considered the fundamental chamber, and showed that this corresponded to the special *subgroups*, and was in fact isomorphic to the power set $\mathcal{P}(S)$. If we exclude the empty simplex, and note that a power set as a poset is isomorphic to itself with the opposite ordering, then we get exactly the nerve L of the Coxeter system, see figure IV.7. If we take the geometric realisation of L as a poset, $|L| = |\mathcal{S}_{>\emptyset}|$, since it is a poset of cells, we get the barycentric subdivision of L (see remark B.2). Looking at the definition of K, it is $|\mathcal{S}|$, so the only vertex added to $|L|$ is the empty simplex, which is a subset of all of the vertices

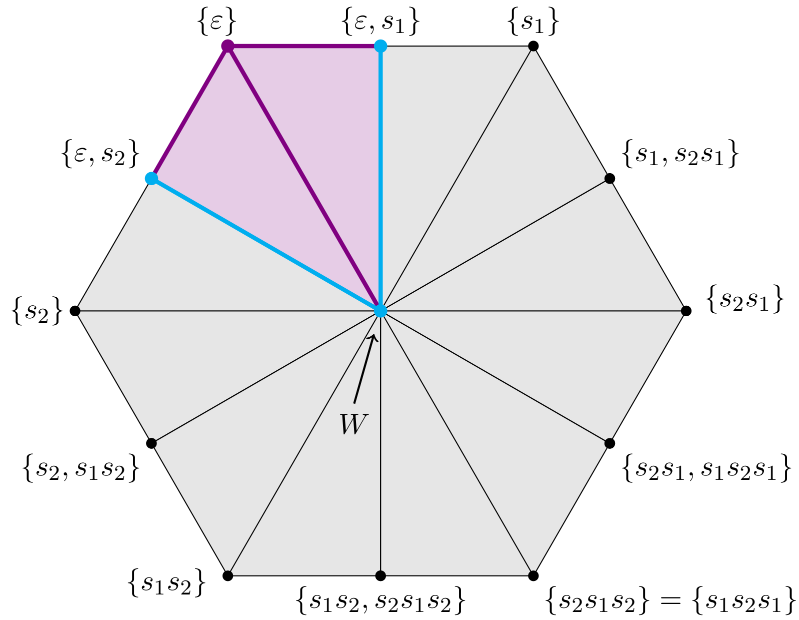


Figure IV.10: The Davis complex Σ of the group D_3, showing the image of K under the inclusion map.

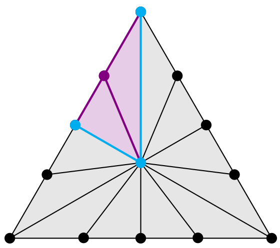


Figure IV.11: The Davis complex of D_3 drawn so as to illustrate the action of D_3 as the symmetry group of the triangle.

of L, so when we take the flag complex, we get the cone over $|L|$, that is $K = \mathbf{Cone}(|L|)$, so the fundamental chamber of Σ is the cone over the barycentric subdivision of the fundamental chamber of X. W acts on the fundamental chamber of X to get the whole of X, and the action of W on the spherical cosets is defined in the same way, so Σ can be obtained by acting W on K. The barycentre of L is S, which gets mapped to W under the inclusion of K into Σ, and this vertex is clearly fixed by the action of W, so Σ is a cone over this point. It follows by a careful analysis[7] of the definitions that in general this gives us that $\Sigma = \mathbf{Cone}(\mathbf{Bs}X)$, as was the case in the previous example.

If W is infinite, one must first remove the part of X corresponding to the infinite special subsets, and then we can play the same game to get Σ. In the particular case that W is infinite, and X is a manifold, the only special subset which is not spherical is S itself (see theorem IV.1), and so only the cone vertex W and all the simplices which have W as a facet, are lost, and hence $\Sigma = \mathbf{Bs}X$. From these observations we get the following results.

[7] Note that the cone over a simplex is a simplex of dimension one higher. With this it can be shown that it is generally the case that K is a cone over both $\emptyset$ and S as in the example, and that removing either of these cone-points leaves isomorphic complexes. So deleting the cone-point S (which is equivalent to deleting the cone-point W in Σ), leaves behind a complex which is isomorphic to the barycentric subdivision of the fundamental chamber of X.

Proposition IV.6. *With notation as above:*

1) *K is contractible. [13, lemma 7.2.5(i)]*
2) *If W is finite and $\#S = n$, then Σ is contractible, moreover $\Sigma \approx D^n$, the closed n–ball.*
3) *If W is infinite and X is a manifold, then Σ is a manifold.*

Proof. (1) K is a cone, and hence contractible. (2) If W is finite, Σ is a cone. Moreover $X \approx \mathbf{Bs}X \approx S^{n-1}$, and so $\Sigma \approx \mathbf{Cone}(S^{n-1}) \approx D^n$. (3) If X is a manifold then $\Sigma = \mathbf{Bs}X$ which is a manifold. ■

At the start of this chapter, we simplified the calculation of X by giving a combinatorial definition. We have given a combinatorial definition of Σ as a flag complex, but the number of steps required to calculate this from W is exponentially greater than the number of steps required to calculate X, since we are dealing with power sets. We shall now look at a quick geometric way to obtain Σ from X (or more precisely from its geometric realisation as a simplicial complex) if W is finite using *Coxeter polytopes*.

Definition IV.10. Let (W, S) be a finite Coxeter system, and consider its reflection representation on a space V. Let x be a point strictly inside the fundamental chamber, then the **Coxeter polytope** is the convex hull W–orbit of x in V. We denote it $\mathcal{C}_W$.

Even more loosely we could just take one point chosen (with common sense) from the interior of each chamber in V, and then $\mathcal{C}_W$ is the convex hull of of these points. In the case of D_3, look at P in figure I.2, there we see that the convex hull of its W–orbit is a hexagon. Since W is finite, $\mathcal{C}_W$ will be homeomorphic to the sphere in V on which W naturally acts. In particular it is easy to see that X triangulates the sphere, and $\mathcal{C}_W$ is the dual[8] of this triangulation. In the case of $W = \triangle(2, 3, 5)$, the symmetry group of the dodecahedron, X is shown in figure I.5b, and $\mathcal{C}_W$ is the dual to this, which is a truncated icosidodecahedron, a polyhedron with faces which are regular decagons, hexagons, and squares[9]. We can then identify Σ with the barycentric subdivision of $\mathcal{C}_W$, indeed this follows by establishing the required relation between $\mathbf{Cone}(\mathbf{Bs}X)$ and $\mathbf{Bs}(\mathbf{Hull}(X^*))$, where X^* is the dual of X, and $\mathbf{Hull}(\cdot)$ is the convex hull.

In the case that the W is infinite, the poset $(W\mathcal{S})_{\leq wW_T}$ is isomorphic to $W_T(\mathcal{S}_T)$, which is the face corresponding to $\mathcal{S}_{\leq wW_T}$. This is isomorphic to the barycentric subdivision of $\mathcal{C}_W W_T$, so if we replace this subdivision with the Coxeter polytope, we shall get a coarser cell structure on Σ which has some very nice properties.

1) The vertex set (0–skeleton) of Σ is W.
2) The 1–skeleton of Σ is the Cayley graph of (W, S), see definition B.19.
3) The 2–skeleton of Σ is the Cayley 2–complex of (W, S).
4) Each cell is a Coxeter polytope.
5) The link of every vertex is isomorphic to $L(W, S)$.
6) The poset of cells of Σ is $W\mathcal{S}$.

The first and fourth are tautological, we shall leave the rest as exercises, they just come from working through the definitions [13, proposition 7.3.4]. With these observations in hand we can improve on the preliminary results above.

[8] The dual of a polyhedron is obtained by replacing each face with a vertex at its centre, joining two of these vertices if the corresponding faces shared an edge, and then each of the original vertices becomes the "corresponding" face of the new polyhedron. The process is analogous in other dimensions.

[9] Identify each of these in figure I.5b; how are they related to the dihedral angles of the triangles?

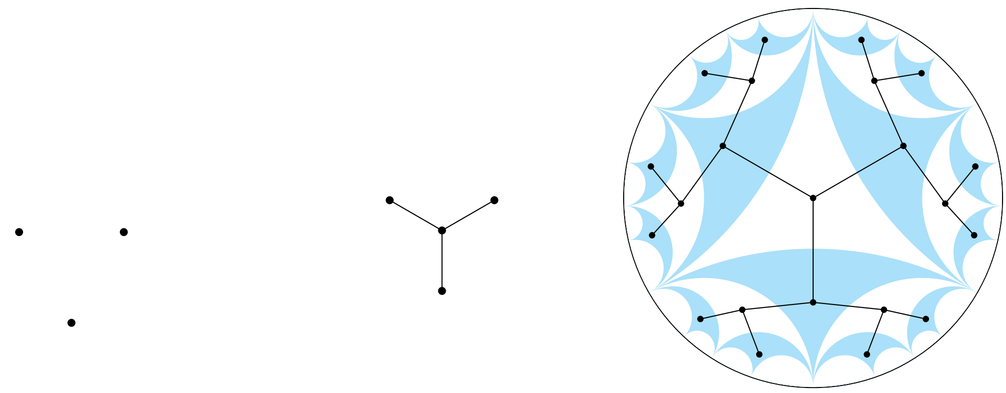
Proposition IV.7. *Σ is simply-connected.*

Proof. A loop in Σ is homeomorphic to a closed loop in its 2-skeleton, which is a Cayley 2–complex, and hence simply connected by proposition B.2. [13, lemma 7.3.5] ∎

Proposition IV.8. *Suppose L triangulates S^{n-1}, then Σ is an n-manifold.*

Proof. This follows from the proof of theorem IV.1 and property 5. [13, proposition 7.3.7] ∎

Example IV.6. In example IV.2 we saw the tiling of the hyperbolic plane by ideal triangles, and in particular we saw that its Coxeter complex of (W, S) was not a manifold. What does its Davis complex look like? $S = \{a, b, c\}$, but $\mathcal{S} = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, and so its nerve $L = \mathcal{S}_{>\emptyset}$ consists of 3 distinct points. Neither of our results giving sufficient conditions for Σ to be a manifold are satisfied. We can calculate Σ explicitly however. We know that $K = \mathbf{Cone}(|L|) = \mathbf{Cone}(Geom(L))$, since $Flag(L) = L$ in this case. K is the fundamental chamber, so we can get Σ by acting on it by W. What we get is shown in figure IV.12. In figure IV.12c, we have shown the cellulation of Σ by Coxeter polytopes, instead of the full complex (which is merely the barycentric subdivision of that shown). It is clear that this is not a manifold; however, if drawn all the way up to the boundary of the hyperbolic plane, we would get the infinite trivalent tree. Looking back at example IV.3, we saw that Σ is in fact a building, with apartments the embeddings of $X(D_\infty, \{s, s'\})$ into Σ.



(a) The nerve of (W, S).

(b) The fundamental chamber of (W, S).

(c) The Davis complex of (W, S), superimposed on the hyperbolic plane.

Figure IV.12

At the start of this section we identified two shortcomings of the Coxeter complex as a space on which to make W act. The first was that the fundamental domain was not compact in general, and we have seen in the construction of K, and with the spherical condition in the definition of Σ, that the W action on the Davis complex admits a compact fundamental domain. The second was that there was no natural W–invariant metric on X. Σ fixes this. If we consider the cellulation of Σ by Coxeter polytopes discussed above, we can embed each in a Euclidean vector space of the same dimension and thus give each a Euclidean metric. This extends to a piecewise Euclidean metric in the whole of Σ in the obvious way (that this is well-defined needs to be proved, but it turns out to be the case). What is more, it was proved by G. Moussong that with this metric, Σ is CAT(0), which means that it is "non-positively curved". The formal definition involves taking a small geodesic triangle and comparing it to a small triangle in Euclidean space, and showing that the original triangle is "thinner", which is to say looks more like a hyperbolic triangle than the Euclidean triangle. It measures the curvature of a geodesic space in some sense[10]. The proof is quite long, and uses a lot of Riemannian geometry beyond the scope of this report[11].

Why is this result so astonishing? It allows us to say that W acts by isometries of Σ, but the highly non-trivial fact is that those isometries are Euclidean. In section 3B we discussed briefly three classes of Coxeter group, what we might call spherical, Euclidean, and (compact) hyperbolic (finite, affine, and hyperbolic), and saw ways in which to make such groups act on those spaces of constant curvature using the reflection representation. In particular we saw that such groups are characterised by the signature of the associated bilinear form B: $(n,0,0)$, $(n-1,1,0)$, and $(n-1,0,1)$ respectively. It is clear from this, given the almost complete freedom we have in writing down a Coxeter matrix, that the vast majority of Coxeter groups are not of any of these types. That Σ with its piecewise *Euclidean* metric can be constructed for *any* Coxeter system is the miracle of this result. This shows why its not surprising that X does not share this property.

Notes

1) The two main works cited in this chapter are [8] and [13]. [25] was also consulted, however the material presented there is written in a style which is inaccessible to someone who has not seen Coxeter complexes before, so where possible we have cited and followed the proofs given by K. Brown.
2) For the discussion leading up to definition IV.4, we have diverged from [8], giving our own explanation which echoes the way we go on to construct Σ, so as to make comparison of the two easier.
3) Except in the case of section 2A, all examples in this chapter are our own. Example IV.3 is mentioned merely in passing in [8]; there it is neither used as motivation for retractions, nor to link to the Davis complex.
4) In [8] mainly weak buildings are discussed. He gives as his first axiom that all the apartments are Coxeter complexes. We have decided to follow [25] here by giving the definition of a thick building and working towards the result that every apartment is a Coxeter complex. Despite this, for the content of the section, we mainly adapt the results in [8] to this more general setting.
5) The only book cited other than [13] which mentions the Davis complex in more than passing is [3], where P. Bahls devotes a section to introducing the very basic ideas, but with no theory. This is a good way to acquaint oneself with the idea before tackling the full construction.
6) We have found no explicit comparison of the definitions of the Coxeter complex and Davis complex, like the one we have provided at the start of section 2C: every book which covers one thoroughly, at most mentions the other in passing. The statement of proposition IV.6(1) is in [13], but the rest of the statement, and the proofs are our own.
7) The proofs of either theorem IV.2, or that Σ admits a W–invariant piecewise Euclidean CAT(0) metric could each be the sole subject of a report such as this. We have aimed to motivate their importance, give sufficient background such that the statements can be understood, and in the case of the former, give a detailed sketch of the argument so that its veracity does not seem unreasonable.

[10] The general definition for a space being CAT(κ) is the same but the comparison triangle is chosen in a space of constant curvature κ, i.e. the sphere or hyperbolic space.

[11] The proof is scattered throughout [13], the theorem statement can be found on p. 235.

Post Script

The aims of this report were three-fold: first to introduce the basic theory of Coxeter groups at a level at which a beginner in the field would be able to understand; second to compare and contrast the two approaches for the subject: geometric and combinatorial; and third to define and introduce the applications of the Coxeter and Davis complexes. We think that one of the most mathematically satisfying things about studying Coxeter groups is the interplay between the many different approaches to thinking about them. In this report we have seen 5 different versions of *chambers*: the regions in V arising from a geometric reflection group, the simplicial cones in V^* of the reflection representation, the maximal simplices of the Coxeter complex X, the translates of the fundamental chamber K in the Davis complex, and most importantly, since the W–action was simply-transitive in all these cases, the elements of W itself, with which we could label the geometric chambers.

With regard to the first aim, we have motivated the study of groups generated by reflections which act discretely. We have studied the way they act on a vector space, and looked at representative range of combinatorial properties, in particular, proving these in depth so as to peer deeply into the structure of the group. We have also been able to illustrate these ideas with many examples . As to the second aim, the approach made to the first three chapters was solely geared towards comparing the geometric and combinatorial view points. We fully justified, as thoroughly as is practical in this setting, the way in which the reflection representation linked these two approaches. Moreover we left no doubt of the efficacy of the geometric approach to solving combinatorial problems. Indeed one might be forgiven for thinking that, after comparing the proofs of propositions A, B, and C using combinatorial and geometric arguments, we would do as well to disregard the combinatorial approach altogether. This however would be a great loss to the field. While geometry is excellent at answering many questions once posed, it is not very good at posing the questions. Indeed none of the results proved would have occurred to us to explore if we merely thought geometrically. The algebra also forms deep connections to other fields of mathematics, in particular Lie groups and Lie algebras. A balance then should be sought between the two approaches.

The third aim was achieved perhaps to a lesser extent than the other two, largely as a result of the pragmatic restraints on the length of the report. In particular we had neither the space nor enough of the relevant background covered to do justice to the Davis complex, as was evident in the slightly rushed treatment of the material at the end. On the other hand, the construction of the Davis complex in the first place is significantly more complex than the construction of the reflection representation, or the Coxeter complex, so we may perhaps at least console ourselves with the hope that the definition has been explained clearly enough that this report could form a solid basis on which to research that direction further. We shall conclude this report with some comments on what areas one might research next and suggested further reading.

One obvious extension to the more classical study of the geometry of Coxeter groups as seen in chapter I is to crystallographic groups. These are the groups of symmetries of lattices, and just as at school one might prove that the only regular tessellations of the plane are by equilateral triangles, squares, and hexagons; one can classify all *Weyl* groups, those Coxeter groups which leave invariant a lattice in the reflection representation V. N. Bourbaki defines a *special point* in V to be a point a through which passes one representative from each of the equivalence classes of $\mathcal{H}$ under the relation of parallelism [6, chapter V, section 3, No. 10]. For finite Coxeter groups, the only special point is the origin, but for affine Coxeter groups, the can form a whole lattice. Identify for example the special points in figure I.6. Studying these sorts of groups naturally leads to the study of root systems, which we have mentioned a couple of times because of their omission. Root systems take the approach of studying essentially the normal vectors corresponding to the hyperplanes instead of the hyperplanes themselves. These are covered in [6, chapter VI] and [9, chapter 4].

In chapter II we studied combinatorial properties of Coxeter groups, and in particular properties of the length function. In section III.2 we mentioned in passing Tits' solution to the word problem which is based in either of the equivalent conditions **(D)** or **(E)**. The full treatment of this combinatorial solution is detailed in many books, for example [8, chapter II, section 3B], or [9, section 4.3]. If one wanted to study the combinatorics seriously, a good source would be [4]. In particular one would look at a partial ordering on a Coxeter group called the *Bruhat ordering*, which is very closely related to the length function. This then leads to the Kazhdan-Lusztig polynomials which were mentioned in the preface.

As explained in chapter IV, the main application of Coxeter complexes is to *buildings*. This report should be sufficient to make the reader very comfortable in approaching [8], which is itself a comfortable introduction to buildings. Here we were not able to do more that have a brief look at some of the material K. Brown covers in his first chapter on buildings. Alternatively, in chapter 5 of [13], M. Davis introduces a very complicated construction of a space $\mathcal{U}$ given a discrete group G, and a *mirrored space* X. This is a significant generalisation of the spaces X and Σ which we defined, and indeed both of these can be defined using $\mathcal{U}$. One can study the homology of this space in general, and with it prove the much stronger version of proposition IV.7, that Σ is contractible [13, theorem 8.2.13], and much else besides. There is also the proof of the existence of the CAT(0) metric on Σ.

For a general introduction to Coxeter groups, we recommend the first 3 chapters of [8], and [17]. For a more formal treatment, the authoritative work on the subject is [6]. This is not for the faint of heart, but very well worth the effort if given the time. [16] also covers much of the material there, but the proofs and exposition are easier to follow. For a detailed study of the combinatorial side of Coxeter groups, one needs go no further than [4]. By the nature of the mathematics, the notation and the proofs are quite heavy, but still approachable. The theory surrounding the material covered here in chapter IV is treated encyclopedically by [1] and [13], which together are more than 1300 pages on the geometry of Coxeter groups and the theory of buildings. We find [1] slightly more approachable than [13], but both cover fascinating material really well.

Appendix A

Spaces on which Coxeter Groups Act

A.1 Classification of Affine Coxeter Groups

Theorem A.1. *Let (W, S) be an irreducible Coxeter system, then it is an affine reflection group if and only if its diagram appears in table E.1. [13, theorem C.1.3]*

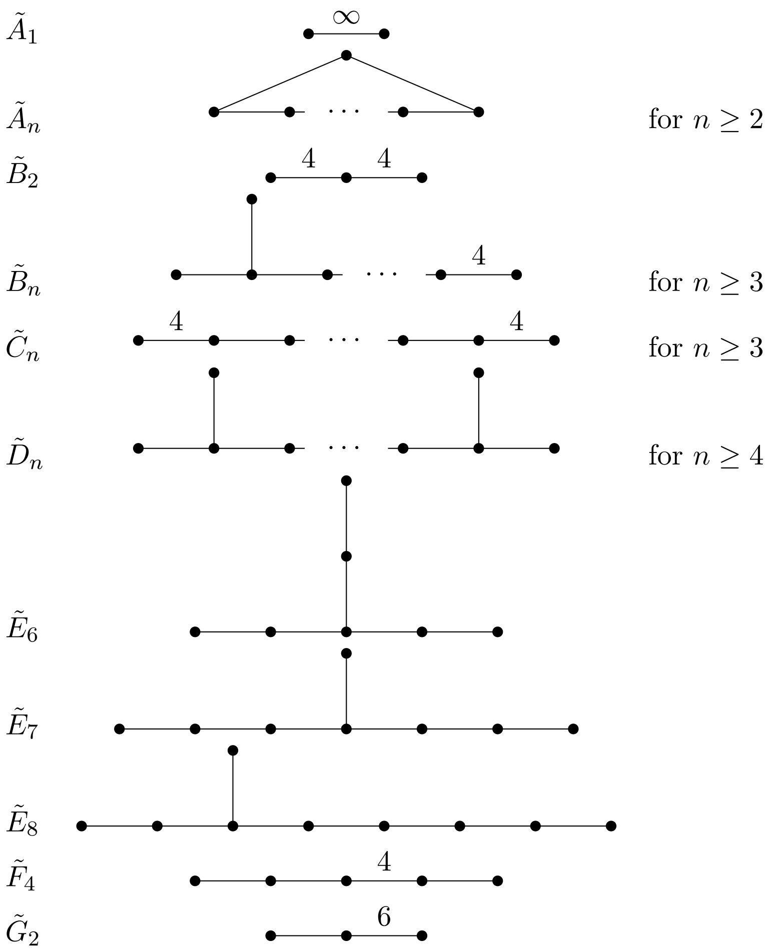


Table E.1: The Coxeter diagrams corresponding to affine Coxeter systems.

A.2 Classification of Compact Hyperbolic Coxeter Groups

Theorem A.2. *Let (W, S) be an irreducible Coxeter system, then it is a hyperbolic reflection group with fundamental chamber which is a compact simplex if and only if its diagram appears in table E.2. [13, theorem C.1.4]*

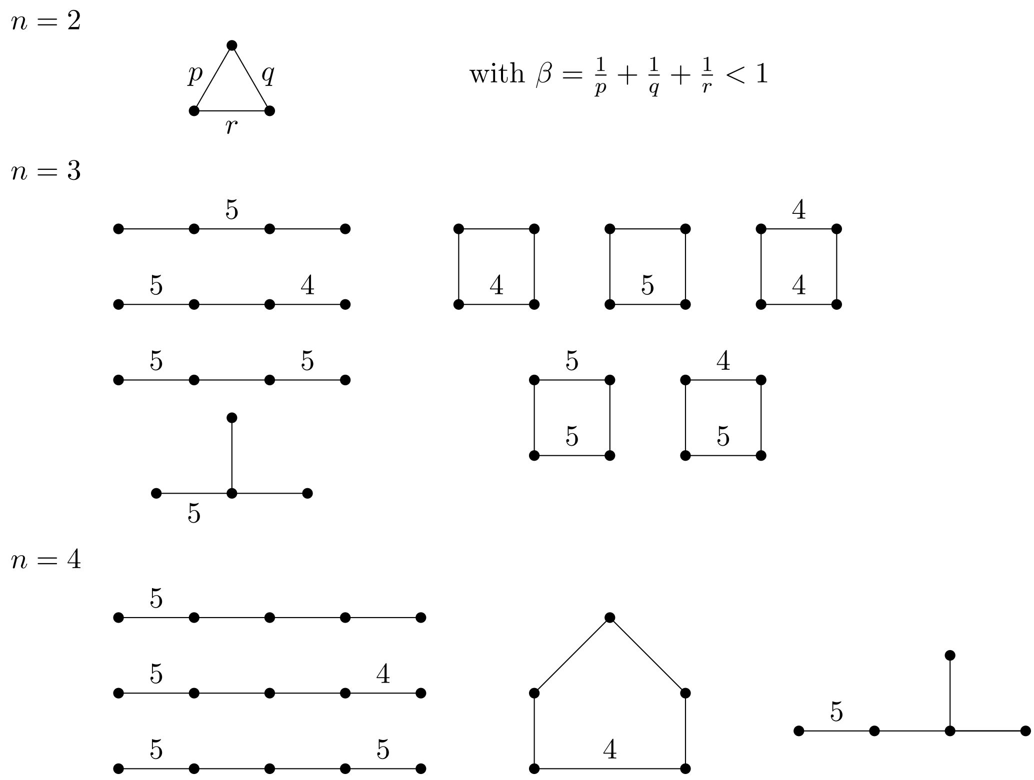


Table E.2: The Coxeter diagrams corresponding to compact hyperbolic Coxeter systems.

A.3 Rigidity and Finite Coxeter Groups

In the proof of theorem III.6, we assumed that W was irreducible in two places: when applying the classification theorem to justify that every Coxeter presentation of a given finite Coxeter group has the same number of generators, and for proving the reverse implication. Without that assumption, we can prove:

Theorem A.3. *If a finite Coxeter group W acts essentially on a vector space V of dimension n then it has a Coxeter presentation with n generators.*

Proof. Assume that W acts essentially on V, then it is clear that W is generated by at least n reflections. By theorem I.2 the associated chambers are simplicial cones. Lemma I.1 says then that each chamber has n walls, and since W is generated by reflections in the walls of any chamber, theorem I.1, W is generated by exactly n reflections (which is a Coxeter-type presentation by the proof that **(A)** *implies* **(C)**, in which the generators which appear in the statement of **(A)** are the same as those in the presentation eventually constructed for **(C)** — for details see the proofs of the various implications either given or cited throughout). ■

The distinction is that a given reducible finite Coxeter group W, it may have presentations with different numbers of generators, say n in one case and m in another, and so W can be made to act essentially on $\mathbb{R}^n$ and on $\mathbb{R}^m$.

This is a manifestation of the Isomorphism Problem. To try and resolve these issues, people study *rigidity* of Coxeter systems, and a survey of these sorts of results is made in [3]. We shall state one result in this area which helps in the case of this particular theorem.

Definition A.1. A Coxeter system (W, S) is **reflection rigid** if there is an automorphism $\alpha \in \mathrm{Aut}(W)$ such that $\alpha(S) = S'$ for every system (W, S') which has the same reflections as (W, S)[1], that is $R = \{wsw^{-1} \mid w \in W,\ s \in S\} = \{ws'w^{-1} \mid w \in W,\ s' \in S'\} = R'$.

The existence of such an automorphism α means in particular that the number of generators of any Coxeter system which has the same reflections must be the same. In such a case one could say that the pair (W, R) has n generators, which is an improvement.

Theorem A.4. *Let (W, S) be a finite Coxeter system, then it is reflection rigid.*

Proof. Let (W, S) be a Coxeter system with R the set of reflections. We claim that the decomposition of W into irreducible components as in theorem II.2 can be reconstructed from the set R. Indeed, take a maximal partition of R such that any two reflections in different elements of the partition commute. The groups generated by the elements of this partition are the irreducible components of W. If (W, S') is another system for W with $R = R'$, then there are isomorphisms between the irreducible factors so we are reduced to the case that (W, S) is irreducible.

Suppose (W, S) and (W, S') are two irreducible systems for W. W is finite, with $R = R'$, so [7, theorem 3.8] says precisely that $\#S = \#S'$. The theorem now follows from the classification of finite Coxeter systems, theorem III.5. [7, theorem 3.10] ■

There are examples of finite Coxeter groups which are not *rigid* (which is what you would need to completely remove the ambiguity over the number of generators), but only reflection rigid. For example the the dihedral groups D_{2k} for k odd [3, p. 53]. One can immediately conclude from the last two theorems that if W is a Coxeter group which acts essentially on two vector spaces of different dimensions, then the set of reflections in those two actions is going to be different, as you would intuitively expect.

[1] A Coxeter *group* is **rigid** if such an automorphism exists for *any* two associated Coxeter systems.

Appendix B

Simplicial Complexes

B.1 Posets

We shall begin be defining some necessary concepts from set theory and topology.

Definition B.1. A set $\mathcal{P}$ with a relation between some of its elements, written $\leq$, is a **poset** (or **p**artially **o**rdered **set**) for all $a, b, c \in \mathcal{P}$

1) $a \leq a$ (reflexive),
2) if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetric),
3) if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitive).

If $p \in \mathcal{P}$, we write $\mathcal{P}_{\leq p} = \{x \in \mathcal{P} | x \leq p\}$. We generalise this in the obvious way to $\mathcal{P}_{<p}$, $\mathcal{P}_{\geq p}$, and $\mathcal{P}_{>p}$

Definition B.2. Let a and b be elements of a poset $\mathcal{P}$; if they exist, the greatest lower bound of a and b is the **meet**, written $a \wedge b$, and the least upper bound of a and b is the **join**, written $a \vee b$.

Example B.1.

1) Let A be any set, then $\mathcal{P}(A)$, the power set of A, is a poset with relation given by set inclusion $\subseteq$.
2) $\mathbb{N}$ is a poset ordered by divisibility; so $1|2, 3$ and $2, 3|6$, but neither $2|3$ nor $3|2$. The meet of 2 and 3 is $2 \wedge 3 = 1$, and their join is $2 \vee 3 = 6$. This is summarised in figure B.1.

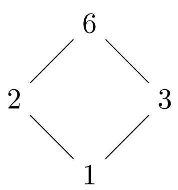


Figure B.1: The Hasse diagram for 6 showing the divisibility relations between it and its divisors.

Definition B.3. Let $\mathcal{P}$ and $\mathcal{P}'$ be posets ordered by $\leq$ and $\subseteq$ respectively. A bijection $\phi : \mathcal{P} \mapsto \mathcal{P}'$ which preserves the order structure, that is $\phi(a) \subseteq \phi(b)$ if and only if $a \leq b$ for all $a, b \in \mathcal{P}$, is called a **poset isomorphism**. If $\mathcal{P}$ and $\mathcal{P}'$ coincide, such an isomorphism is called an **automorphism**.

B.2 Abstract Simplicial Complexes

Definition B.4. An **abstract simplicial complex** consists of a (possibly infinite) vertex set $\mathcal{V}$ and a collection Δ of finite subsets of $\mathcal{V}$ called **simplices**, such that

1) $\forall v \in \mathcal{V}$, $[v] \in \Delta$,
2) if $\sigma \in \Delta$ and $\sigma' \subseteq \sigma$, then $\sigma' \in \Delta$.

where we write simplicies with square brackets. We shall sometimes require that the empty simplex is included, so that any two simplices have a lower bound, and hence a meet, we shall sometimes specifically exclude the empty set[1]. We say the **dimension** of $\sigma \in \Delta$ is $dim(\sigma) := \#\sigma - 1$, then σ of dimension k is a k–**simplex**, and $\sigma' \subseteq \sigma$ in Δ is a **facet** of σ. The **dimension** of Δ is the maximum over the dimension of the simplicies in Δ. We say that the empty simplex has dimension -1.

A subset Δ' of Δ is a **subcomplex** if it is an abstract simplicial complex. For k less than or equal to the dimension of Δ, the k–**skeleton** of Δ is the subset $\Delta^{(k)}$ of all simplicies of dimension at most k.

For a normal simplicial complex, we think about triangles and line segments embedded in some real space. In this case however, we are abstracting away almost everything, and saying that having specifying 3 vertices is as good as specifying a triangle, hence a 3–simplex becomes just a 3 element subset of an arbitrary set.

Remark B.1.

1) The first condition specifies that indeed all the vertices in the vertex set are 0–simplicies in the abstract simplicial complex, so $\Delta^{(0)} = \mathcal{V}$.
2) The second condition just means that if Δ contains some k–simplex, then it also contains all of its facets.
3) Clearly Δ is a poset ordered by set inclusion, and if $\sigma \in \Delta$, then $\Delta_{\leq\sigma} = \mathcal{P}(\sigma)\backslash\{\emptyset\}$ or $\mathcal{P}(\sigma)$.

Definition B.5. Let Δ and Δ' be abstract simplicial complexes. A map $\psi : \Delta \mapsto \Delta'$ is a **simplicial map** if it maps vertices to vertices and simplices to simplices. If the image of any simplex under ψ is a simplex of the same dimension, we say that ψ is **non-degenerate**. Any such non-degenerate simplicial map is a poset isomorphism when it is restricted to any simplex in Δ.

In the preceding comments, we made some effort divorce the abstract simplicial complex from the geometric intuition about ordinary simplicial complexes. Now we shall undo that. Given an abstract simplicial complex, we can realise it as a simplicial complex in a real vector space. We do exactly what one might expect: if $\mathcal{V}$ is the vertex set take $\#\mathcal{V} = n$ vectors (in general position) in $\mathbb{R}^{n-1}$ of appropriate dimension, labelled by $\mathcal{V}$. Then if $\sigma \in \Delta$ is the k–simplex $[v_1, \ldots, v_{k+1}]$, add a k–simplex to $\mathbb{R}^{n-1}$ with the appropriate vertices.

More precisely, if Δ is an abstract simplicial complex with vertex set $\mathcal{V}$, let $\Delta^{\mathcal{V}}$ be the standard ($\#\mathcal{V}$)–simplex whose vertices are labelled by $\mathcal{V}$. For each finite non-empty $\sigma \in \Delta$, denote by ρ_σ the face of $\Delta^{\mathcal{V}}$ spanned by σ.

[1] This awkwardness is unavoidable. In his work on Coxeter complexes and buildings, J. Tits included the empty simplex in his definitions, and this has some non-trivial consequences: we would have to amend axiom (B4) for buildings, and we use the empty simplex tacitly throughout section 1D of chapter IV. See remark IV.1 for more discussion of the interpretation of the empty simplex. On the other hand M. Davis specifically excludes it when defining for example, the nerve of a Coxeter system, see definition IV.7.

Definition B.6. The **geometric realisation** of Δ is $Geom(\Delta)$, which is the subcomplex of $\Delta^{\mathcal{V}}$ defined by

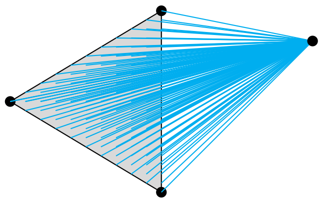
$$\rho_\sigma \in Geom(\Delta) \iff \sigma \in \Delta$$

B.3 Join

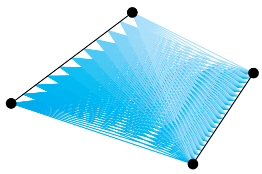
Definition B.7. The **join** of abstract simplicial complexes J and K with vertex sets $\mathcal{V}_J$ and $\mathcal{V}_K$, can be defined as the subcomplex of $\mathcal{P}(\mathcal{V}_J \sqcup \mathcal{V}_K)$ (the full abstract simplicial complex on $\mathcal{V}_J \sqcup \mathcal{V}_K$) given by

$$\begin{aligned} \rho_\sigma \in J \vee K \iff \#\sigma \geq 2 \text{ and } \sigma \text{ conains exactly one element of} \\ \mathcal{V}_J \text{ or exactly one element of } \mathcal{V}_K. \end{aligned}$$

This is equivalent to the definition of join for general posets given above. Taking the geometric realisation of the complexes, $Geom(J) \vee Geom(K) = Geom(J \vee K)$ where the join of two simplicial complexes consists of the two complexes embedded disjointly in the same space, along with the union of every line between every point in ever simplex of J and every point in every simplex of K, see figure B.2.



(a) The join of a simplex and a point forms the cone of that simplex.



(b) The join of two intervals is a tetrahedron.

Figure B.2: Two examples of joins of simplicial complexes.

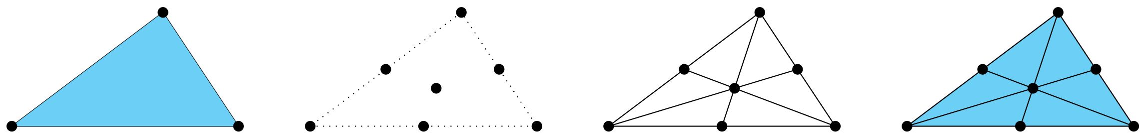
Definition B.8. Let Δ be an abstract simplicial complex, v a disjoint point, then we shall write $\mathbf{Cone}(\Delta) = \Delta \vee [v]$ for the **cone** of Δ over v. If no v is specified, we shall assume that the point is chosen arbitrarily.

We shall tend not to bother with saying *abstract* simplicial complex, merely simplicial complex. As one can get between abstract simplicial complexes and geometric simplicial complexes, it does not make much of a difference, but we shall generally be thinking in abstract terms.

B.4 Barycentric Subdivision

Barycentric subdivision is the natural way of dividing up a simplicial complex into more simplicies. We shall give an informal definition, motivate this definition, and then give the formal definition.

Take a simplicial complex Δ in $\mathbb{R}^n$, and for every simplex σ in Δ, choose a point x_σ in the interior of σ, called the **barycentre** of σ; in particular you might choose x_σ to be the "centre of mass" of σ. Now you add edges to all of the nearby barycentres (noting that each vertex is its own barycentre, so we still have all of the original vertices), and fill in higher dimensional simplices in accordance with the original simplicial complex, see figure B.3.



(a) (b) (c) (d)

Figure B.3: The barycentric subdivision of a 2–simplex (a). The barycentres are shown in (b), the edges to the nearby barycentres in (c), and the full barycentric subdivision in (d).

Why might you want to do such a thing? Suppose you have a continuous map between two compact topological spaces; it would be nice if we could approximate this map by a simplicial map — the simplest kind of map. Suppose one has a triangulation for the spaces. If the triangulation of the first space is very efficient in the number of simplicies it uses, then one could easily imagine coming up with some continuous map which was not compatible with the triangulations (try mapping a triangle continuously onto a square via a simplicial map — its clearly not possible, even though both are homeomorphic to a circle). However, by subdividing the first complex sufficiently, you will create enough simplicies that the continuous map is compatible with the triangulations. The way we formalise this is using the barycentric subdivision. That you can do this for any continuous map is called the *simplicial approximation theorem* [2, Section 5.5, theorem 2].

Definition B.9. Let Δ be an abstract simplicial complex with vertex set $\mathcal{V}$. The **barycentric subdivision** of Δ, written $\mathbf{Bs}\Delta$, is the simplicial complex whose vertex set is Δ, and $\tilde{\sigma} = \{\sigma_1, ..., \sigma_k\} \subseteq \Delta$ is a simplex in $\mathbf{Bs}\Delta$ if $\sigma_1 \subseteq \cdots \subseteq \sigma_k$, i.e. σ_1 is a facet of σ_2 which is a facet of σ_3 and so on. We denote the n^{th} barycentric subdivision of Δ by $\mathbf{Bs}^n\Delta$.

How do we relate this definition to our previous picture? Before we chose a point x_σ in the interior of each simplex to be the barycentre of that simplex, so we have one barycentre for each simplex in the original complex. In the formal definition therefore, we just choose the vertex set (which is an abstract set) to be exactly the set of simplices in the original complex. The next step was to build up the simplices in the new complex from the old one. Think about what being one of the "nearest" barycentres means in terms of the facet relation between simplices. Similarly how should the 2–simplices in $\mathbf{Bs}\Delta$ relate to the simplices in Δ in terms of the facet relation? It is easy to convince yourself that the condition we have put on $\tilde{\sigma}$ for it to be a simplex in $\mathbf{Bs}\Delta$ is the correct one.

Barycentric subdivision can be extended to cells which are not simplicies in the analogous way, see figure B.4 on page 85 for the barycentric subdivision of a hexagon.

B.5 Flag Complexes

Definition B.10. An **incidence[2] relation** on a set $\mathcal{V}$ is a binary operation which is symmetric and reflexive.

If $\mathcal{V}$ is a set equipped with an incidence relation, a **flag** in $\mathcal{V}$ is a subset of pairwise incident elements. We write $Flag(\mathcal{V})$ for the set of all finite flags in $\mathcal{V}$, partially ordered by inclusion.

Definition B.11. Since an incident relation is reflexive, $[v] \in Flag(\mathcal{V})$ for all $v \in \mathcal{V}$. If $\sigma \in Flag(\mathcal{V})$, its elements are all pairwise incident, hence any subset $\sigma' \subseteq \sigma$ is also a finite flag. Hence $Flag(\mathcal{V})$ is an abstract simplicial complex, which we call the **flag complex** of $\mathcal{V}$.

[2] An incidence relation is an equivalence relation where we drop the requirement that it be transitive. The name comes from geometry, where we say two lines are incident if the share a common point. This is clearly reflexive and symmetric, but the case of two parallel lines in the plane, with a third line making an angle with these two, contradicts transitivity.

Remark B.2. Given a poset $\mathcal{P}$, we say that $p, q \in \mathcal{P}$ are incident if either $p \leq q$ or $q \leq p$. So the flags in $\mathcal{P}$ are the finite chains in $\mathcal{P}$, that is to say, the finite totally ordered subsets of $\mathcal{P}$. Suppose now that $\mathcal{P}$ is in fact the poset of cells of an abstract simplicial complex Δ. Comparing the above description with definition B.9, we see that $Flag(\Delta) = \mathbf{Bs}\Delta$.

We say that a collection of simplices in a simplicial complex is **joinable** if they have a join (in the sense of definition B.2) *in* the simplicial complex. We can now give a very simple characterisation of when a simplicial complex Δ is a flag complex.

Proposition B.1. *Let Δ be a simplicial complex, then the following are equivalent:*

1) *Δ is a flag complex,*
2) *Every finite set of pairwise joinable simplices is joinable.*
3) *Every set of three pairwise joinable simplices is joinable.*
4) *Every finite set of pairwise joinable vertices is joinable.*

Proof. The equivalence of (1), (2), and (4) is obvious, so too is the fact that (2) *implies* (3). The final implication (3) *implies* (2) follows by simple induction. [8, chapter I, appendix B, proposition 1] ■

Definition B.12. The **geometric realisation** of a poset $\mathcal{P}$ is the geometric realisation of the flag complex of $\mathcal{P}$, $Flag(\mathcal{P})$. We write

$$|\mathcal{P}| = Geom(Flag(\mathcal{P}))$$

We have two very different but related definitions of geometric realisation, the first of a simplicial complex, the second of a poset. In example IV.5 we have a poset which is also a simplicial complex, and we shall consider both the geometric realisation of it as a simplicial complex, and as a poset, and see that they are different.

B.6 Chamber Complexes

Definition B.13. A finite dimensional simplicial complex Δ is a **chamber complex** if all maximal simplicies have the same dimension (equivalently, every simplex of dimension less than Δ is a facet of a simplex of higher dimension), and can be connected by a **gallery** (a sequence of maximal simplicies in which consecutive simplicies share a face, i.e. are **adjacent**). We shall call the maximal simplices **chambers**.

Definition B.14. A chamber complex is **thin** if every co-dimension 1 facet is the face of exactly two chambers, it is **thick** if every co-dimension 1 facet is the face of at least 3 chambers.

Definition B.15. Let Δ and Δ' be two chamber complexes of the same dimension, then a non-degenerate simplicial map ψ from Δ to Δ' is called a **chamber map** (i.e. a simplicial map which takes chambers to chambers, see definition B.5). If Δ' is a subcomplex of Δ, and $\psi|_{\Delta'}$ is the identity map, then ψ is called a **retraction**.

Chamber maps necessarily take adjacent chambers to adjacent chambers, and hence galleries to galleries.

B.7 Colourings

Definition B.16. Let Δ be a chamber complex of finite dimension n, and let I be an abstract set with n elements. A **colouring**[3] of Δ by I is a map which assigns to each vertex an element from I such that the vertices of each chamber have distinct "colours". The **type** of a simplex on Δ is the subset of I used to colour its vertices.

Lemma B.1.

1) *If a chamber complex is colourable, then there is only one colouring up to bijections between the colouring sets.*
2) *If a chamber complex is the barycentric subdivision of a simplicial complex, then it is colourable.*

Proof.

1) If a colouring of one chamber is fixed, then since the adjacent chambers share all but at most one vertex with the original chamber, their colouring is also determined. Since all chambers can be connected by a gallery, this colouring extends uniquely to the whole complex.
2) Since the vertices of a barycentric subdivision are the simplices in the original complex, one can colour them by their dimension.

[8, p. 30] ■

We can rephrase the definition of a colouring in terms chamber maps. Let Δ be a simplicial complex Δ^I be the simplex with vertex set I, for I a set of size $\dim \Delta + 1$. A colouring of Δ is the same as a chamber map $\kappa : \Delta \mapsto \Delta^I$; hence Δ is colourable if and only if it admits such a chamber map.

B.8 Chamber Systems and Links

Definition B.17. Suppose Δ is a colourable chamber complex, coloured by a set I. Any co-dimension 1 simplex is labelled by $I \backslash \{i\}$ for some $i \in I$. For a fixed $i \in I$, two chambers are i–**adjacent** if they share a face which is coloured by $I \backslash \{i\}$. This is an equivalence relation for each i, and so we define the **chamber system** associated to Δ to be the collection of chambers of Δ together with the relations of i–adjacency.

Definition B.18. Let Δ be a chamber complex and A a simplex of Δ. The **link** of A in Δ, written $lk_\Delta A = lkA$ is the set of simplicies which do not have A as a facet but are nevertheless joinable to A. The link is the boundary of the smallest subcomplex containing a small open neighbourhood (once translated into the geometric realisation) of A in Δ.

lkA is a subcomplex of Δ. The maximal simplicies of lkA are in bijective correspondence with the chambers which have A as a facet. In figure B.4, the link of the central vertex is the boundary of the hexagon for example. lkA may not itself be a chamber complex. For example, suppose removing A and all its facets from Δ left it disconnected, then lkA is not connected, so not every maximal simplex would be joined to every other by a gallery. In figure B.4, the link of the central vertex is the boundary of the hexagon for example.

We can use the link to say when we can safely forget about everything else in Δ, and just think about its chamber system.

[3] K. Brown uses the term "labelling" in [8] which we avoid so as not to confuse with our labelling of chambers by elements of W. Indeed it would seem that he came to the same idea, as this is the terminology adopted in [1].

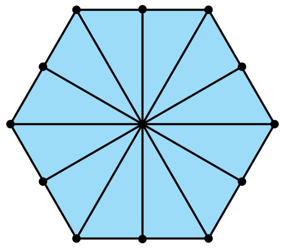


Figure B.4: A chamber complex, the barycentric subdivision of a hexagon.

Lemma B.2. *Let Δ be a coloured chamber complex in which the link of every vertex is again a chamber complex; then Δ is determined up to isomorphism by its chamber system.*

Proof. We say that two chambers in a chamber system are $(I\backslash\{i\})$–equivalent if they can be joined by a gallery in which no two consecutive chambers are i–equivalent. This is another equivalence relation. The condition pertaining to links means that under this relation, two chambers are $(I\backslash\{i\})$–equivalent if and only if their vertices coloured i coincide.

We can use this to reconstruct Δ, it has one vertex coloured i for each $(I\backslash\{i\})$–equivalence class. A subset of these vertices is a simplex in Δ if and only if their corresponding equivalence classes have non-empty intersection. [8, chapter 1, appendix D, proposition 1] ■

B.9 Cayley Graph and Complex

Definition B.19. Let G be a group, with a set of generators S each of which is order 2. The the **Cayley graph** $Cay(G, S)$ has vertex set G, and there is an edge between g and g' if and only if $g' = gs$ or $g = g's$ for some $s \in S$.

Since S generates G, $Cay(G, S)$ is connected, G clearly acts on $Cay(G, S)$ as a graph automorphism by left multiplication, and moreover G acts simply-transitively on the vertices. Since the identity is not in S, $Cay(G, S)$ is a simple graph (i.e. it contains no double edges or cycles of length 1). Given the Cayley graph of a group G, S can be read off by looking at the nearest neighbours of some chosen vertex, which can be thought of as the identity by simple-transitivity.

Definition B.20. Let G be a group with presentation $\langle S \mid \mathcal{R}\rangle$, and let $Cay(G, S)$ be the corresponding Cayley graph. The **Cayley presentation 2–complex**, written $\Lambda = Cay(G, \langle S \mid \mathcal{R}\rangle)$ is the 2–complex obtained from $Cay(G, S)$ by adding a 2–cell for every relation in the group generated by $\mathcal{R}$ excluding relations of the form s or s^2, for $s \in S$ (where we write all relations in $\mathcal{R}$ in the form $word = \varepsilon$, and then generate a group by the words on the left hand side). Words in $\langle\mathcal{R}\rangle$ are sequences of generators, which correspond to closed loops in $Cay(G, S)$ from any chosen base vertex, so we add a polygonal cell which has boundary one of these loops, starting from each vertex of $Cay(G, S)$ to get Λ.

Proposition B.2. *Λ is simply connected.* ■

For a proof, see [13, proposition 2.2.3]

Notes

Sections B.1 to B.4 are based on [13, appendix A], and sections B.5 to B.8 are based on [8, chapter I, appendix]. Section B.9 is based on [13, chapter 2]. Consultation of [3] has also been made.

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The main sources cited in this report are [3], [6], [8], [9], [13], [16], and [17]. Although [1] is based on [8], we have, where possible, referenced the latter as we have studied this in detail, but used the other only for reference.

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